CHAOTIC STIRRING OF PASSIVE FLUID BY A VORTEX PAIR IN CIRCULAR DOMAIN

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Summary The advection problem of a passive fluid in 2D velocity field induced by two point vortex pair in an inviscid incompressible fluid inside a circular domain is considered. It is shown that the motion of fluid particles can demonstrate chaotic properties which result under certain initial conditions in an intensive stirring regime. Various criteria and methods were proposed in order to identify these regions, including: an analysis of phase trajectories, spectral and correlation analyses, construction of Poincaré sections, calculation of largest Lyapunov exponents. Some additional criteria are offered, namely: definition of the changes of length of border of investigated contours in time, analysis local stretching maps of passive contours and some statistical methods adapted to 2D fluid flows. Comparative analysis of various stirring criteria and methods in a velocity field induced by two and three point vortices in a circular domain is presented.

The stirring of a passive fluid domain with arbitrary borders in known velocity field induced by two point vortices is discussed. This problem has relations with various practical applications in chemical, pharmaceutical and food industries, and also in ecology. On the other hand, mixing phenomenon represents a fundamental scientific problem closely connected to the modern concepts of chaotic and regular dynamics [1].

It is known that laminar flows can result in intensive stirring regimes under certain conditions. Such regimes, which usually are called in the literature as chaotic ones, are a subject of intensive research both from theoretical, and from the experimental points of view. The main importance in these researches is that the velocity field in some laminar flows can be presented in analytical forms.

An advection problem is limited to the analysis of the trajectories of Lagrangian fluid particles (markers), which form borders of the region under investigation, in Eulerian velocity field. Every fluid particle can be treated as a passive fluid particle governed by vector equation

$$d\mathbf{x}/dt = \mathbf{U}(\mathbf{x},t), \qquad \mathbf{x}(0) = \mathbf{x}^0. \tag{1}$$

Here $\mathbf{x}^0(t)$ is the position of fluid particles (markers) and $\mathbf{U}(\mathbf{x},t)$ is the velocity field. The investigation is reduced to the study of an evolution of fluid particles in the flow. Every such particle moves along its own trajectory, and the equations (1) can predict particle positions at any moment. If we analyse evolution of an arbitrary contour presented by system of markers, then ordered connection of markers provides the boundary of the domain for a given moment of time.

The goal of the present study is a comparative analysis of various criteria of stirring processes in 2D bounded domains. We illustrate our analyses and main results using an example of fluid flow generated by N point vortices in a circular cavity. The achieved results allow to formulate the main peculiarities of stirring processes of a passive impurity.

Let us consider a system of N point vortices of intensities k_α at positions (x_α, y_α), $\alpha = 1,...,N$, in a circular cavity of radius a. Motion of vortices described by [2]

$$k_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \qquad k_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha}, \qquad H = -\frac{1}{4\pi}\sum_{\alpha=1}^{N} k_\alpha \left[\sum_{\beta=1}^{N}{}' k_\beta \ln r_{\alpha\beta} - \sum_{\beta=1}^{N} k_\beta \ln \rho_{\alpha\beta} \right], \tag{2}$$

where $r_{\alpha\beta}^2 = (x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2$, $\rho_{\alpha\beta}^2 = (x_\alpha - a^2 x_\beta / R_\beta^2)^2 + (y_\alpha - a^2 y_\beta / R_\beta^2)^2$, $R_\beta^2 = x_\beta^2 + y_\beta^2$.

The introduction of the stream function Ψ permits to present equations (1) for every markers with coordinates (X, Y) in Hamiltonian form

$$\frac{dX}{dt} = \frac{\partial \Psi}{\partial Y}, \qquad \frac{dY}{dt} = -\frac{\partial \Psi}{\partial X}, \qquad \text{with} \qquad X(0) = X^0, \qquad Y(0) = Y^0 \tag{3}$$

where
$$\Psi = -\frac{1}{2\pi}\sum_{\alpha=1}^{N} k_\alpha \ln \frac{(X - x_\alpha)^2 + (Y - y_\alpha)^2}{(X - a^2 x_\alpha / R_\alpha^2)^2 + (Y - a^2 y_\alpha / R_\alpha^2)^2}.$$

Here the prime denotes omission of the singular term $\alpha = \beta$.

Consider an example of a periodic motion of two point vortices (vortex pair, $k_1 = -k_2 = k = 1.0$) initially located at $x_1^0 = 0.7$, $x_2^0 = -0.3$ and $y_1^0 = y_2^0 = 0.0$ inside a circular domain of radius $a = 1.0$. Here and in what follows we use non-dimensional length with a and non-dimensional time scale at a^2 / k. In this case vortices move periodically with a period $T = 23.53$. Fig.1 shows trajectories of vortices, and fig.2 illustrates the Poincaré section for passive fluid markers.

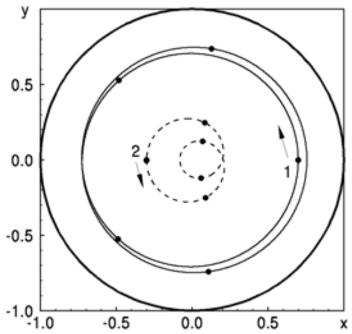

Fig.1 Trajectories of vortices.

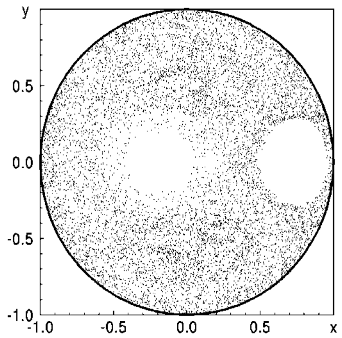

Fig.2 Poincaré section for marker.

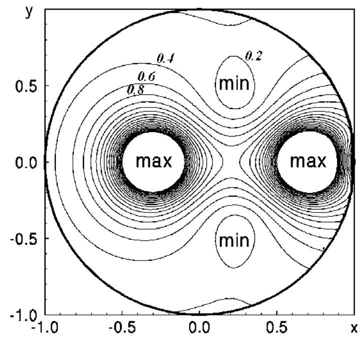

Fig.3 Local stretching map for T=0.

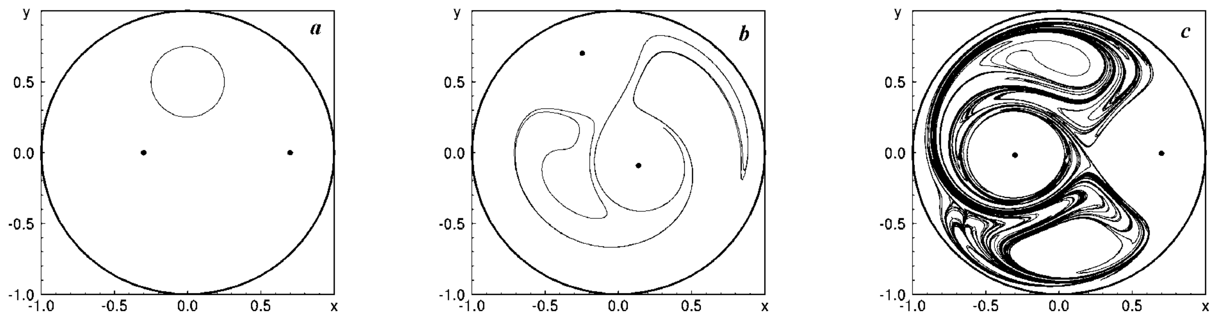

Fig4. Deformation of a passive contour by two point vortices at: (a) $t = 0$, (b) $t = T$, and (c) $t = 3T$.

Although two vortices have regular trajectories, the movement of passive markers in a given velocity field can be both regular and chaotic. The Poincaré section shows that the flow contains two regular zones located near point vortices. The fluid particle located in other part of circular domain, participates in chaotic motion. The similar results give other criteria, namely: spectral and correlation analysis of trajectories, values of the largest Lyapunov exponent for chosen markers both from regular and from chaotic zones of Poincaré section.

However, stretching of contour borders in the chaotic regions of marker motion demonstrates different stirring properties. Local stretching maps [3], which show a degree of exponential stirring of fragments of the contour placed at a point (x, y) of given velocity field for the default moment, and testify that those parts of a contour, which have an intensive stretching, are located near point vortices. There are even regions in the chaotic zone of motion, in which stretching appears insignificant (the region marked “min” in fig. 3). It is interesting to note that zones of moderate stirring are inside of chaotic motion of markers identified by Poincaré section (fig. 2).

Let us consider a deformation of a passive fluid contour formed by circular domain of radius $r_0 = 0.25$ with the centre placed at $(0.0, 0.5)$, fig.4a. Vortices move in counter-clockwise direction. Positions of border contour, which correspond to moments $t = T$ and $t = 3T$, are plotted in figs.4b and 4c, respectively. The comparative analysis shows that in the top and bottom parts (in order to the initial vortex position) the flow forms two boundary regions. These zones are near the wall, where the normal component of velocity field is equal to zero, and form a region of an intensive stirring, which is approximately in the central part of the circular domain.

The comparative analysis of criteria shows that there is no universal criteria among a wide set of criteria of regular and chaotic motion of passive fluid particles. Each of the considered criteria is directed to the analysis of the certain peculiarities of evolution. It is possible to conclude that it is necessary to use a complex of criteria to detect the stirring regime. In some cases using separate criterion can result in erroneous results.

References

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