CHAOTIC ADVECTION IN A MIXER WITH CHANGING GEOMETRY

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Summary Chaotic advection is examined in a mixer consisting of a circular vat full of fluid, stirred by an arbitrary number N of stirring rods. The fluid is assumed to be highly viscous, and the flow two-dimensional. A series solution for the velocity field permits an extremely precise computation of the paths of passive fluid particles under motion of the stirring rods. Of particular note is the case $N > 2$, which allows the generation of 'topological chaos' provided the stirring rods move with appropriate topology. To date, the stirring rods have had circular cross-section; it is shown here how the series-solution approach can be modified to accommodate other cross-sectional profiles, and that stirring can be made more effective using elliptical paddles rather than circular ones.

INTRODUCTION

Over the past twenty years or so [1, 2], it has become well established that fluids can be stirred effectively even in simple two-dimensional laminar flows. The key to effective stirring in such flows is the generation of chaotic paths for fluid particles. The position $(x(t), y(t))$ of a fluid particle is given by $\dot{x} = u(x, y, t)$, $\dot{y} = v(x, y, t)$, where (u, v) is the fluid velocity, and particle paths can readily be made chaotic if the time-dependence of the velocity field is chosen appropriately.
The first, pedagogical examples of flows exhibiting the chaotic advection of fluid particles were inviscid flows, driven by various forms of singularities (vortices, sources, sinks) – see, for example, [1]. Soon afterwards the opposite limit, of Stokes flow in a highly viscous fluid, was investigated, for example the flow in an eccentric annular space, driven by successive rotations of each boundary. The essential motivation behind the choice of either inviscid or highly viscous fluids was the desire for an exact expression for the fluid velocity field. Such an expression could generally be computed in all the cases studied. With an exact expression for $u(x, y, t)$ and $v(x, y, t)$, it is possible to carry out numerical integrations of fluid particle advection with extreme precision, and this is a prerequisite to tracking faithfully the particle paths (a significant computational inconvenience is that of course errors grow exponentially in a chaotic flow).
Most of the viscous-flow devices that possess an exact expression for the velocity field contain no genuinely moving parts – time-dependence in the flow is generated by motion of the boundaries of the flow domain parallel to themselves in time-dependent fashion. A consequence of this restriction is that the geometry of the device remains fixed, which is not ideal in a model for a mixer, but, crucially, permits an exact solution to the problem. One exception to this rule of fixed geometry is the 'TRM' [3], which models slow viscous flow in a vat of circular cross-section driven by the motion of a single stirring rod, also of circular cross-section. The streamfunction is readily derived for general motion of the stirring rod, and, in fact, can be constructed from a superposition of results from the fluid and solid mechanics literature dating back to the 1940's. The stirring rod in the TRM is driven by a planetary gear and it is a straightforward matter to choose the gear ratios so as to provide a highly effective stirring protocol. This is achieved largely by trial and error, and careful tuning of the system parameters.

STIRRING WITH PADDLES

Although the TRM has a genuinely time-dependent geometry, it has only one stirring rod. Clearly one would like to have at one's disposal an arbitrary number (N, say) of stirring rods, moving simultaneously through the vat, rather than only one. The case $N > 1$ provides a more realistic model for any industrial planetary mixer. However, it seems unlikely that an exact expression for either inviscid flow or Stokes flow could be found in the case of general N. But for the purposes of numerically tracking the evolution of fluid particles it is not the existence of an exact expression for (u, v) that matters, just the means to compute the velocity with extreme (preferably machine) accuracy. Fortunately, a series solution for the streamfunction of the motion can be developed [4, 5], which gives very rapid convergence and sufficient accuracy that it may be considered effectively 'exact'. The coefficients in the series are obtained numerically by minimizing the residual squared error in the boundary conditions.

Non-circular paddle cross-section

The very rapid convergence of the series is undermined if one attempts to model stirring rods whose cross-section is other than circular. However, we show that the method is readily adapted to accommodate simple paddles, provided one can find a conformal mapping $w(z)$ from the paddle to a circle. The series are then written in terms of the mapping function w rather than in z. For example, elliptical paddles are easily modelled, since the requisite conformal map is well known, and by varying the aspect ratio of the ellipse we span the spectrum of cases from circular paddles to flat paddles. We have been able to simulate paddles down to aspect ratios of around 10^{-7} with no computational difficulties.
The method of solution is highly efficient. Since an exact expression is posed for the streamfunction (and similar expressions for the velocity components are readily obtained), all that needs to be computed and stored is the corresponding set of coefficients of the sums involved. It is also a simple matter to compute various other quantities of physical interest, such as the energy usage of the device.

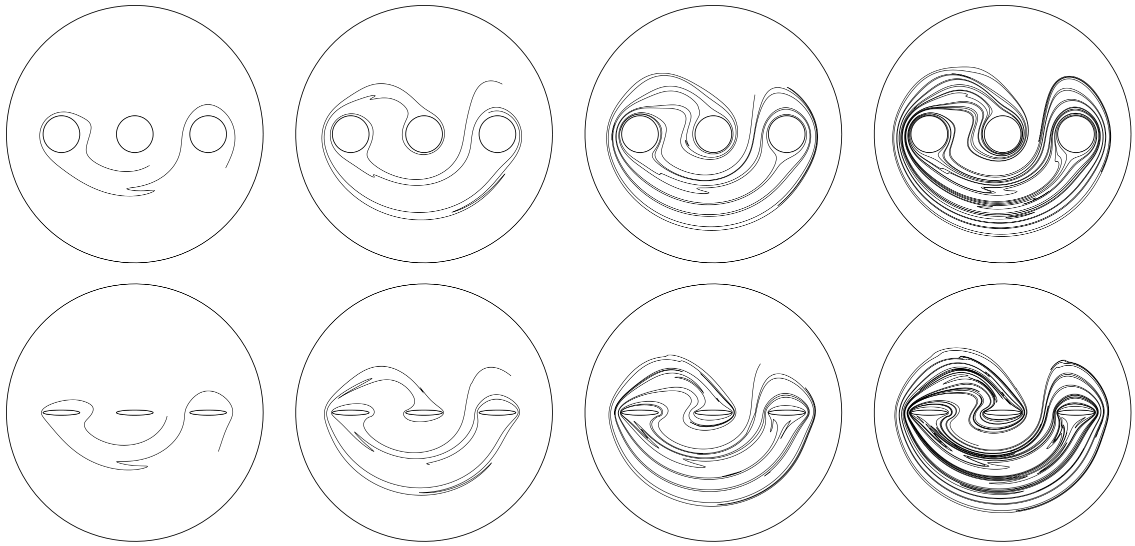

Figure 1: Top: evolution of a line element, stirred by circular paddles. Bottom: as top, but with elliptical paddles of aspect ratio 0.14. The line element grows by a factor 95.3 (top) or 123 (bottom), indicating greater stretching by the elliptical paddles.

Topological chaos

Not only does a mixer with multiple stirring rods accord more closely to everyday notions of a mixing device and to industrial planetary mixers, it also permits the generation of 'topological chaos' [6] – if the stirring rods are braided around one another in a topologically nontrivial way, there is a region of the flow domain in which a certain degree of chaos is guaranteed. Striking experiments by Boyland, Aref and Stremler [6] have shown that careful consideration of the topology of the motion of three circular stirring rods leads to a dramatic improvement in the stirring effectiveness, and that superficially similar stirring protocols can generate very different results. Using our series solution we have simulated their experiments [4] and find excellent qualitative agreement.

Results

We have simulated topological chaos in a batch mixer with $N = 3$ stirring rods generating Stokes flow in a circular vat. The rods are moved according to the following protocol: first the right-hand pair are interchanged, in a counterclockwise motion, then the new left-hand pair are interchanged, but clockwise. The evolution of a line element whose length is initially equal to the radius of the vat is shown in Figure 1, after one, two, three and four applications of this protocol. One might expect flat paddles to stir rather better than circular ones: the figure illustrates results for both circular and elliptical paddles, and shows that, at least in the specific case considered, this is indeed the case.

CONCLUSIONS

Stirring of a highly viscous fluid by an arbitrary number of stirring rods of arbitrary cross-section is readily described with high accuracy, sufficient for the tracking of chaotic fluid particle paths. The solution method easily accommodates elliptical paddles of aspect ratio ranging from unity (circular paddles) to extremely small values (e.g. 10^{-7}, very flat paddles). Topological chaos is readily simulated and we find some empirical evidence that elliptical paddles stir the fluid more effectively than circular ones.

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