NON-UNIQUE QUASI-EQUILIBRIUM TURBULENT BOUNDARY LAYERS

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Summary An asymptotic investigation of turbulent boundary layers having a moderately large velocity defect is presented. It extends the classical small-defect theory insofar as the defect is measured by a second perturbation parameter besides the sufficiently large global Reynolds number. Most remarkably, the theory is capable of describing non-uniqueness of quasi-equilibrium flows, a property which has been discussed intensively in the literature.

MOTIVATION AND PROBLEM FORMULATION

Near-equilibrium turbulent boundary layers play an important role in internal flow situations. In particular, here diffuser flows serve as a typical engineering application. In such a flow configuration the boundary layer has to sustain a preferably large pressure rise, exerted by the external irrotational bulk flow, whereas the flow shall remain strictly attached. In order to prevent separation it is advisable to control the pressure gradient such that the boundary layer globally admits a self-preserving state, or, equivalently, remains close to equilibrium. As a well-established condition necessary for self-similarity, the streamwise velocity component at the boundary layer edge must vary as a power m of the streamwise coordinate. Moreover, it has been commonly argued on a partially empirical basis that $m > -1/3$, cf. [4].
On the other hand, it is known from experiments that $m \approx -0.22$ characterises a boundary layer exhibiting identically vanishing wall shear ([3]). As a consequence, this suggests that (at least) double-valued solutions are possible as the associated exponent m measuring the strength of the pressure gradient may fall below that critical value but is greater than $-1/3$. Indeed, in the past several studies indicated the non-uniqueness of turbulent near-equilibrium boundary layers ([3], [4]). An early hint is given by Clauser in his pioneering experimental work [1]. Most notably, an explicit clue is found in [5]: In that study the impact on diffuser design of boundary layer flow that withstands a pressure increase much larger than one provoking separation was investigated numerically using an integral method.
However, this striking feature of near-equilibrium flow has not been investigated so far by a strict rational approach based on first principles. It is, among others, the primary objective of our presentation to elucidate this particular flow structure by means of an asymptotic analysis of the Reynolds-averaged Navier–Stokes equations in the limit of a large global Reynolds number *Re*. To this end, first the few basic assumptions underlying the 'classical' theory of self-preserving boundary layers for $Re \to \infty$ have to be summarised briefly.

ASYMPTOTIC THEORY OF SELF-SIMILAR BOUNDARY LAYERS

The 'classical' limit

A rational asymptotic description of high Reynolds number wall-bounded turbulent shear flows has been formulated first in the early 1970ies, see e.g. the seminal paper of Mellor [2]. It can be shown that this self-consistent theory effectively exploits well-known dimensional arguments which determine the scaling of the viscous sublayer adjacent to the wall and is based on a two-layer structure. Inside the viscous wall layer the streamwise velocity component is of $O(1/\ln Re)$. Matching with the results holding in the fully turbulent main layer yields the celebrated universal logarithmic law of the wall. Moreover, it requires that the velocity defect with respect to the imposed external streamwise velocity U_e as well as the boundary layer thickness δ, non-dimensional with a global length scale, are asymptotically small and also of that magnitude. As a result, the theory does not cover separating flows which apparently exhibit a velocity defect of $O(1)$.
It is demonstrated in [3] on basis of this classical approach that equilibrium flows are characterised by a Rotta–Clauser parameter which varies only slowly in streamwise direction x. Let y denote the distance normal to the wall, it is given by

$$\beta(x) = -\frac{\delta^* U_e \,\mathrm{d}U_e/\mathrm{d}x}{\tau_w} = O(1), \quad \text{with} \quad \delta^* = \int_0^\delta (1 - u/U_e)\,\mathrm{d}y. \tag{1}$$

Furthermore, one infers from the leading-order integral momentum balance that $\beta \sim -m/(1+3m)$. Consequently, the classical theory ceases to be valid as $m \to -1/3_+$ since it predicts an unbounded growth of both the velocity defect and the boundary layer thickness. In present literature this failure is commonly attributed to incipient separation ([3]). In contrast to this suggestion it is shown here that this failure can be avoided by a generalized small-defect theory which is based on a three- rather than a two-layer structure. While the velocity defect is still small, it is, however, asymptotically large compared to the one assumed in the classical case.

Distinguished limit $\beta^{3/2}/\ln Re = O(1)$

In the new theory the double limit $\beta \to \infty$, $Re \to \infty$ considered is found to implicate a wake-type flow in the outermost layer which in fact closely resembles a separating flow. Here the defect is measured by a second small parameter besides $1/\ln Re$ which, among others, characterises the slenderness of the boundary layer. In contrast to the classical analysis,

the new one accounts for weakly nonlinear effects due the inertia terms in the equations of motion. Introduction of the coupling parameter $\Gamma \propto \beta^{3/2}/\ln Re = O(1)$ leads to a new distinguished limit where the still finite wall shear enters the flow description in second order. There the solution is shown to satisfy a solvability condition derived from the integral momentum balance. Assuming strict equilibrium up to second order, the resulting algebraic relationship can – without adopting any turbulence closure – be cast into canonical form,

$$9\hat{D}^2\hat{\mu} = 1 + \hat{D}^3, \quad \text{with} \quad \hat{D} \propto \Gamma^{1/3}, \tag{2}$$

which clearly reveals a double-valued flow structure, see the dashed curve in figure 1 (a). Here $\hat{D}$, and $\hat{\mu}$ measure the velocity defect and the small deviation of the exponent m, characteristic of the external flow, from $-1/3$, respectively. From (2) the fundamental conclusion can be drawn that for high but finite values of *Re* the effects caused by nonlinearities indeed imply a restriction $m > -1/3$, where $m + 1/3 \propto \hat{\mu}/(\ln Re)^{2/3}$.

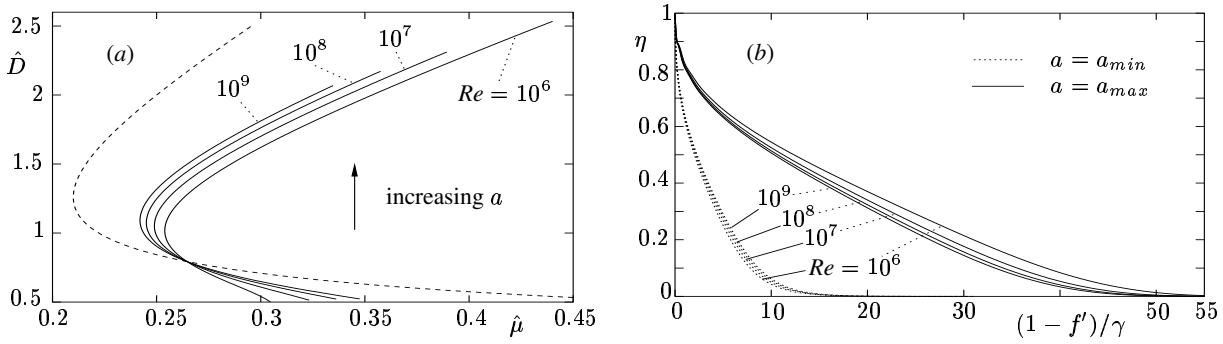

Figure 1. (a) Defect measure $\hat{D}$ (canonical scaling), (b) defect profiles (classical scaling) for different values of *Re*, $a_{min} \le a \le a_{max}$.

Comparison with numerical results for finite values of *Re*

In order to support the asymptotic results a numerical study was performed by solving the nonlinear boundary layer equations assuming strict equilibrium in the fully turbulent flow regime. By introduction of a suitably defined stream function $f(\eta)$ and by applying an appropriate similarity transformation, they become

$$\begin{gathered} m(f'^2 - 1) - (1+m) f f'' = \tau'/a, \quad \eta = y/\delta, \;\; a = \mathrm{d}\delta/\mathrm{d}x = const; \\ \eta \to 0: \;\; f' \sim (\gamma/\kappa)\ln(\eta\, a\gamma Re), \;\; \tau \sim \gamma^2, \;\; f \sim \eta f', \quad \eta = 1: \;\; f' = 1, \;\; \tau = 0. \end{gathered} \tag{3}$$

Here $\kappa \approx 0.42$ denotes the v. Kármán constant. The boundary conditions for $\eta \to 0$ reflect the behaviour of the flow in the overlap regime between the outer region and the viscous wall layer. To solve the resulting problem (3), it is supplemented with an (asymptotically correct) algebraic closure for the Reynolds shear stress τ. Prescribing a (sufficiently large) Reynolds number, a properly chosen (sufficiently small) linear increase of δ denoted by a as well as a minimum value of η, the exponent m is regarded as an eigenvalue and is thus part of the solution. This procedure allows to calculate the expected double-valued velocity distributions, plotted in figure 1 (b), for a given pressure gradient.

From a rigorous rational point of view (3) represents an *ad hoc* approximation of the full set of Reynolds equations. The Reynolds number enters the solution solely via the logarithmic near-wall portion of the flow. As a consequence, the asymptotic error is inherently of $O(\gamma)$. Therefore, extending the domain of the calculations to the boundary $\eta = 0$ by adopting a wall layer model would not improve the quality of the numerical solution in the limit $Re \to \infty$.

By integration of (3) from $\eta = 0$ to $\eta = 1$ one recovers the well-known v. Kármán's integral momentum balance, specified for equilibrium boundary layers. Introducing a canonical equilibrium shape factor $\hat{G} = O(1)$, it is written as

$$\beta(1+3m) = -m + (1+2m)\hat{D}^3, \quad \text{with} \quad \hat{D}^3 = \gamma\beta^{3/2}\hat{G}, \tag{4}$$

where the Rotta-Clauser parameter defined by (1) has to be calculated numerically. The solid curves in figure 1 (a) represent the associated solutions for $\hat{D}$ which show qualitatively good agreement with the predictions of the asymptotic analysis. There the Reynolds number enters in the form $1/\ln Re$, hence the collapse of the numerical results for finite values of *Re* onto the dashed line holding in the limit $Re \to \infty$ is rather slow. Finally, we note that the lower branches in figure 1 (a) reveal the classical results for $\hat{\mu} \to \infty$. In contrast, the upper branches indicate the existence of a fully nonlinear theory predicting a velocity defect of $O(1)$ and, in turn, even separated flows. This issue is a topic of current research.

References

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