EXTENDING THE GENERALIZED LOGARITHMIC LAW TO THE WALL

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Summary Physical arguments for the Reynolds number dependence of the mean velocity profile of canonical turbulent boundary layer with zero pressure gradient are discussed. Based on these ideas the classical two-layer approach is extended and a generalized log law is derived. This new law consists of a logarithmic and exponential term. While the first term covers part of the usual overlap region, the second term describes the inner part of the wake zone. All parameters appearing in the new law are Reynolds number dependent. The new law is extended toward the wall employing a mixing length approach that shows the correct $y^{+\,3}$-dependence of the Reynolds stress close to the wall.

PHYSICAL ARGUMENTS

For several years a new debate concerning the mean velocity profile of canonical turbulent boundary layers with zero pressure gradient (ZGTBL) has taken place. Under this type of flow, a fully developed turbulent boundary layer which is free of any effects from roughness, wall curvature and outer turbulence is understood. Mostly experiments are interpreted to confirm or reject one or another theory. Therefore, we will start here with a short physical argument on why a Reynolds number dependence of the mean profile of a ZGTBL exists and persists until Re reaches infinity.

First Argument: Horse shoe vortices which are connected with the wall reach far into the outer region of the boundary layer and fluid structures from the outer region dive deep into the boundary layer [1]. If viscous structures reside in the so-called inertial sublayer of the ZGTBL, then such layer does not exist in reality.

Second Argument: The quantity that appears in the momentum equation is the Reynolds shear stress gradient and not the Reynolds shear stress itself. The momentum balance for channel and pipe flow is:

$$-\frac{1}{\rho}\frac{dp}{dx}+\frac{d\tau}{dy}+\nu\,\frac{\partial^2 U}{\partial y^2}=0 \qquad \text{with } \tau=-\overline{u\,v} \tag{1}$$

At the position where the shear stress has its maximum, the second term $d\tau / dy$ of eq. (1) becomes zero. In the vicinity of this point, pressure force is only balanced by viscous force. The conclusion is that at least in channel and pipe flow viscous forces are of importance in a region that has been thought to be inertial [2].

Third Argument: Instead of a definition, mostly a syndrome is used to characterize a flow as turbulent. This turbulence syndrome consists of several symptoms among them the energy cascade describing the energy transport from turbulence production to dissipation [3]. This process can be observed in all y-regions of a ZGTBL meaning that throughout the entire boundary layer friction is of importance to maintain this energy cascade.

All three arguments together crumble the arguments for a simple log law with constant parameters. A more sophisticated mean velocity profile can be either derived based on first-principles employing Lie-group theory [4], or by extending the classical two-layer approach to higher-Reynolds-number dependent terms [5].

THE GENERALIZED LOG LAW

Traditionally the ZGTBL is split into an inner and an outer layer. While in the inner zone friction dominates, in the outer layer inertia is more important. It is assumed that each can be described with an asymptotic expansion.

$$\text{inner layer:}\quad u^+\sim\sum_{i=0}^{n}\gamma_i\left(\delta^+\right)u_i^+\left(y^+\right) \qquad\qquad \text{outer layer:}\quad U\sim\sum_{i=0}^{n}\Gamma_i\left(\delta^+\right)U_i\left(\eta\right) \tag{2}$$

Here u^+ and y^+ denote the non-dimensional velocity u/u_τ and coordinate $y\,u_\tau/\nu$ of the inner layer, U is the non-dimensional velocity $(u-u_\delta)/u_\tau$, and η the non-dimensional coordinate of the outer layer, $\gamma_i(\delta^+)$ and $\Gamma_i(\delta^+)$ are gauge functions depending on the Kármán number $\delta^+ = \delta\,u_\tau/\nu$. We generalize an idea of Afzal [6] and obtain after a considerable amount of algebra the generalized log law for the inner layer follows:

$$u^+=\sum_{i=0}^{n}\frac{1}{\delta^{+^i}}\left(\frac{1}{\kappa_i}\ln\left(y^++D_i\right)+G_{1,i}\exp\left(-G_{2,i}\,\eta\right)\right) \tag{3}$$

For $\delta^+ \Rightarrow \infty$ and $\eta \Rightarrow 0$ the simple log law with an additional constant inside the logarithm is obtained. Due to this constant the generalized log law does not show a straight line in the semi-logarithmic plot. For simplicity we assume that in the vicinity of the wall the exponential term in eq. (3) is close to 1 and $D_i = D_1$ and after rearranging eq. (4) follows.

This relation signifies that the parameters of the generalized log law κ and B are Reynolds number dependent (for details see [5]). Note that the Kármán number is strongly related with the momentum thickness Reynolds number Re_Θ.

$$u^+ = \sum_{i=0}^{n} \frac{1}{\delta^{+i}} \frac{1}{\kappa_i} \ln\left(y^+ + D_1\right) + \sum_{i=0}^{n} \frac{1}{\delta^{+i}} G_{1,i} = \frac{1}{\kappa\left(\delta^+\right)} \ln\left(y^+ + D_1\right) + G_1\left(\delta^+\right) \tag{4}$$

MIXING LENGTH APPROACH FOR THE GENERALIZED LOG LAW

To extend eq. (4) to the wall, we employ a mixing-length approach originally introduced by Chapman & Kuhn [8] for the simple log law. This mixing-length approach was reformulated (5a) to ensure that eq. (4) is satisfied. Taylor series expansion (5b) at $y^+ = 0$ shows that the new approach has the correct y^{+3} - dependency in the vicinity of the wall.

$$l^+ = \kappa\left(\delta^+\right)\left(y^+ - D_1\right)\left(1 - \exp\left(-\frac{y^+}{\lambda^+\left(\delta^+\right)}\right)\right)^{\frac{3}{2}} \quad \text{(5a)}$$

$$-\overline{uv} = \frac{D_1^2\, \kappa\left(\delta^+\right)^2 y^{+3}}{\lambda^+\left(\delta^+\right)^3} + O\left[y^{+4}\right] = A\left(\delta^+\right) y^{+3} + O\left[y^{+4}\right] \quad \text{(5b)}$$

Analysing ZGTBL data from Österlund [7] the parameters for the mixing-length approach were derived. It is found that while κ (fig. 1), G_1 and the damping factor λ^+ depend on the Kármán number the factor A (fig. 2) is nearly constant and has a value of $6 * 10^{-4}$ which is close to $4 - 8 * 10^{-4}$ given in [3].

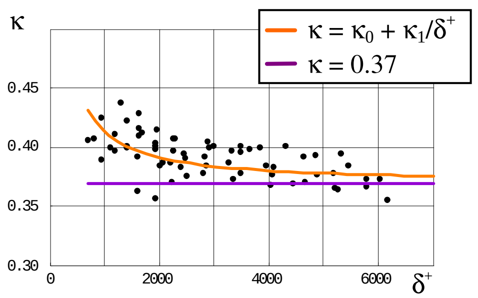

Fig. 1: Kármán “constant” for generalized log law

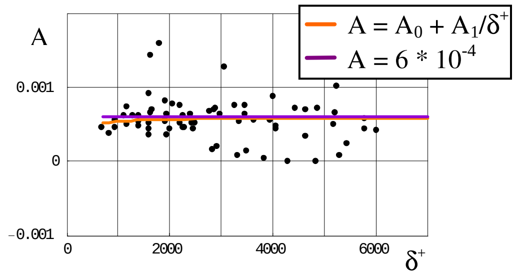

Fig. 2: Parameter A from eq. 6

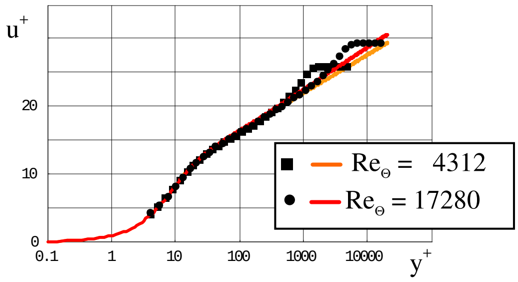

Fig. 3: Mean velocity profile (full line: calculation, symbols: experiments from Österlund [7])

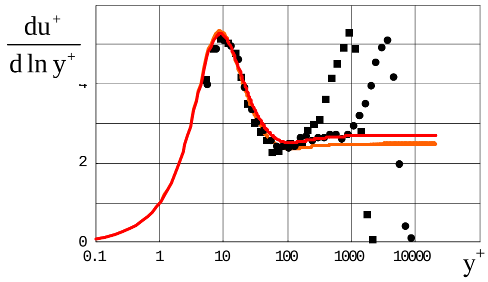

Fig. 4: Normalized gradient of mean velocity profile (curves and symbols as fig. 3)

CONCLUSIONS

Physical arguments which explain the Reynolds number dependence of the mean velocity profile of a turbulent boundary layer are discussed. Based on these ideas the two-layer approach is extended and a generalized log law is derived. A mixing-length approach originally introduced for the simple log law is reformulated to extend the new log law to the wall. Taylor series expansion shows that the new mixing-length approach has the correct y^{+3}-dependence close to the wall.

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