THE DEVELOPMENT (AND SUPPRESSION) OF VERY SHORT-SCALE INSTABILITIES IN BUOYANT BOUNDARY LAYERS

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Summary The flow over a heated (or cooled) flat plate is shown to develop very short-scale instabilities independent of Reynolds number. These are in the form of non-modal, algebraically, disturbances. Their existence presents a number of challenges for the numerical simulation of such boundary-layer flows.

The equations governing the boundary-layer flow over a heated (or cooled) surface are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \qquad \frac{\partial p}{\partial y} = G_0 T, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \\ u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2}, \end{aligned} \tag{1}$$

with boundary conditions

$$u = v = T - T_w(x) = 0 \quad \text{at} \quad y = 0, \qquad u - 1 = T = p = 0 \quad \text{as} \quad y \to \infty. \tag{2}$$

Here (u, v, T, p) represents the basic velocity, temperature and pressure fields within the boundary layer, σ is the Prandtl number and $G_0 = Gr Re^{-1/2}$ is a buoyancy parameter, $y = Re^{-1/2} y^*$ is the boundary-layer variable with Re the Reynolds number and Gr the Grashof number of the flow (see [2] for full details). In this study the buoyancy parameter G_0 is taken to be order unity, and thus the flows under consideration can be classified as mixed forced-free convection boundary layers.

Recently Denier *et al.* [1] demonstrated that self-similar solutions of the boundary-layer equations (1) can support algebraically growing instabilities whose growth rates are independent of Reynolds number. In this case the boundary-layer field variables can be written in the form

$$(u, T, p) = (u_0(\eta), g_0(\eta), q_0(\eta)) + x^{\lambda}(u_1(\eta), p_1(\eta), q_1(\eta)) + O(x^{2\lambda}), \tag{3}$$

where $\eta = y/\sqrt{2x}$ is the similarity variable. The numerical solution of the eigenvalue problem for the growth rate yields positive values for λ for $G_0 < 0$. An asymptotic analysis of the limit $G_0 \to 0^-$ demonstrates that $\lambda \to \infty$ in this limit (full details can be found in [1]).

Although these results for self-similar flows are intriguing they assume particular algebraic forms for the surface temperature which provide little insight into the flow over a heated surface that may be experimentally realisable. However for non-self similar flows, non-modal instabilities still arise and they give rise to some serious difficulties with regards the numerical solution of the boundary-layer equations (1). Figure 1 presents a plot of the spatial development of the wall-shear stress for the flow over a flat plate with prescribed wall temperature given by $T_w(x) = e^{-x} + \gamma(1 - e^{-x})$, where γ is treated as a parameter. The results in figure 1 are for different values of the streamwise step-size Δx; we see that, irrespective of the size of Δx, the results suffer a spontaneous breakdown which is characterised by the appearance of sudden oscillations (closer inspection shows that these oscillations are of a streamwise point-to-point nature). This behaviour is not dependent upon the plate far-downstream being cooled (as is the case when $\gamma = -0.1$) as evidenced by figure 1(b). Furthermore no such breakdown is found for larger values of γ, as can be seen in figure 1(c), where the solution proceeds downstream with a far-downstream form being asymptotically approached.

To investigate the breakdown encountered in the cases $\gamma = 0.1$ and -0.1 we adopt a procedure which is closely analogous to that described above for the similarity states. We seek local solutions of the form

$$(f, g, q) = (f_0(\eta; x), g_0(\eta; x), q_0(\eta; x)) + \epsilon(f_1(\eta; x), g_1(\eta; x), q_1(\eta; x)) \exp\Theta(x) + O(\epsilon^2), \tag{4}$$

where the amplitude ϵ is assumed small. Writing $\lambda = x\Theta_x$, assuming that the basic flow (f_0, g_0, q_0) and the disturbance amplitudes (f_1, g_1, q_1) are slowly varying in x, the boundary-layer equations give, at $O(\epsilon)$, an eigenvalue problem identical to that which arises for the similarity forms. The results for this eigenvalue problem are presented in figure 2.

It is immediately apparent that in both cases, a large (infinite) eigenvalue forms at a finite downstream location, which therefore suggests that infinitely short wavelength disturbances are responsible for the numerical marching

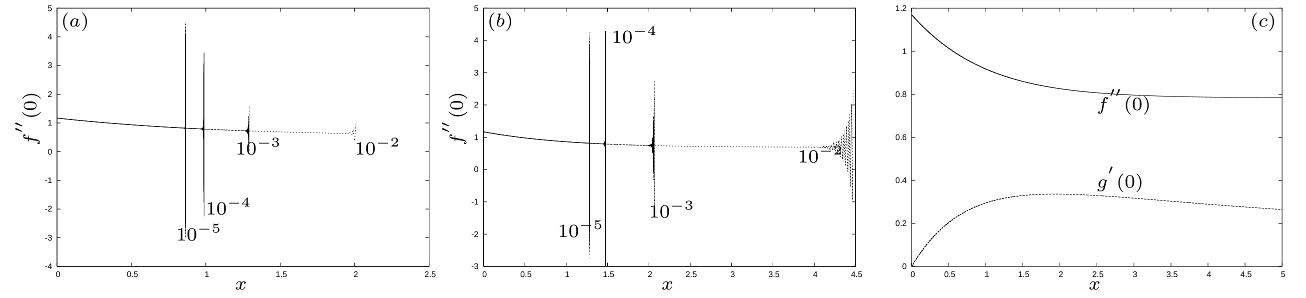

Figure 1: Spatial development of wall-shear stress for the case $G_0 = 0.5$, Δx as shown. Cases (a) $\gamma = -0.1$, (b) $\gamma = 0.1$ and (c) $\gamma = 0.25$ (wall temperature gradient also shown).

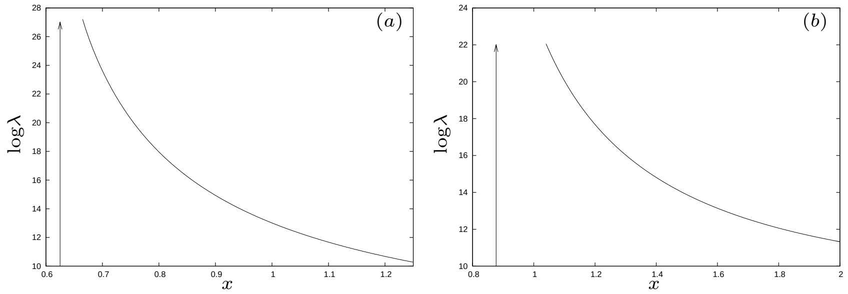

Figure 2: Downstream variation of local eigenvalues for the case $G_0 = 0.5$, (a) $\gamma = -.1$ (b) $\gamma = .1$

difficulties experienced with $\gamma = -0.1$ and $\gamma = 0.1$. A similar eigenvalue searching procedure was adopted for the case $\gamma = 0.25$, but this failed to detect any positive values of λ, an observation entirely consistent with the lack of difficulties encountered with the marching scheme in this case. Therefore in cases where these infinitesimal wavelength unstable disturbances exist, marching schemes will (inevitably) fail. Indeed, the results shown in figure 2 clearly indicate that downstream the wavelength of the disturbances increases, and it is therefore entirely reasonable to conclude that these will only be detected by numerical marching schemes when the numerical grid is of sufficient resolution to detect these disturbances.

The difficulties associated with the failure of marching schemes are a serious restriction on the usefulness of the procedure. However these instabilities can be suppressed by treating the (parabolic) boundary-layer equations (1) in a semi-elliptic manner by imposing (physically reasonable) downstream as well as upstream boundary conditions. This type of procedure, in the context of parabolic systems, can be justified, insofar as it 'selects' the appropriate eigen-form to give the desired behaviour to the problem downstream. The results of just such a calculation, employing Neumann boundary conditions at a finite x location downstream, are shown in figure 3. The complete suppression of the small-scale instabilities is clearly observed.

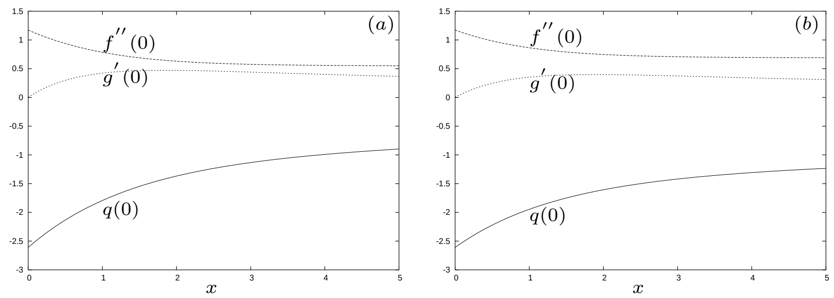

Figure 3: Spatial development of wall-quantities for the case $G_0 = 0.5$, (a) $\gamma = -.1$, (b) $\gamma = .1$

References

[1] Denier, J.P., Duck, P.W. & Li, J., 2004 On the growth (and suppression) of very short-scale disturbances in mixed forced-free convection boundary layers. *J. Fluid Mech.* (submitted).

[2] Denier, J.P. and Mureithi, E., 1996 Weakly nonlinear wave motions in a thermally stratified boundary layer, *J. Fluid Mech.* **315**, 293–316.