DYNAMO ACTION IN STEADY HELICAL PIPE FLOW

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Summary The steady, pressure-driven flow of a conducting fluid down a helical pipe of rectangular cross-section is shown to drive a kinematic dynamo at moderate values of the magnetic Reynolds number, R_m. The asymptotic structure of the growing modes is analysed as $R_m \to \infty$. This is the first laminar, pressure-driven dynamo to be found.

INTRODUCTION

A velocity field giving rise to spontaneous magnetic field generation in a conducting fluid is known as a *kinematic dynamo*. Various simple examples of kinematic dynamos are known, but there are very few dynamically self-consistent examples, in which the initial instability grows to a level where it interacts with the driving mechanism. In this paper we consider configurations with helical symmetry, as originally formulated by [1] and [2]. The Ponomarenko [3] kinematic dynamo can be regarded as a special case of helical flow. Here, a fully nonlinear laminar dynamo driven merely by a steady pressure gradient along a pipe is described. The pipe has a helical shape with a rectangular cross-section, as shown in figure 1. The work has relevance to the construction of laboratory dynamos [4]. Further details can be found in [6].

HELICALLY SYMMETRIC PIPE FLOW

Helical symmetry is a natural generalisation of two-dimensionality ($\varepsilon = 0$) and axisymmetry ($\varepsilon \to \infty$.) In terms of cylindrical polar coordinates (r, θ, z), a scalar function is helically symmetric if it depends only on r and $\phi = \theta + \varepsilon z$, where ε is a constant. The symmetry direction is designated by the Beltrami vector field, $\mathbf{H}$, which is related to the unit coordinate vectors $\mathbf{e}_\theta$ and $\mathbf{e}_z$ by

$$\mathbf{H} = (\mathbf{e}_z - \varepsilon r \mathbf{e}_\theta)/h^2 \qquad \text{where} \qquad h = (1 + \varepsilon^2 r^2)^{1/2} \qquad \text{and} \qquad \nabla \wedge \mathbf{H} = -\frac{2\varepsilon}{h^2}\mathbf{H}.$$

The Navier-Stokes equations and the magnetic induction equation are invariant with respect to this symmetry, and thus helically symmetric solutions to both can be found. The helically symmetric incompressible flow $\mathbf{u}$ can be conveniently represented by two scalar functions $v(r, \phi)$ and $\psi(r, \phi)$ as

$$\mathbf{u} = v\mathbf{H} + \mathbf{H} \wedge \nabla\psi\,.$$

The governing equations for ψ and v are geometrically linked.

Steady, pressure-driven laminar flow down a helical pipe was calculated in [5] as a function of the hydrodynamic Reynolds number, R_e. A solution is shown in figure 2 for an intermediate value of R_e. The two left-hand diagrams show the contours of ψ and v, the cross-pipe and down-pipe flow. An important feature of the flow is that the cross-pipe component has a stagnation point structure, with a weaker, counter-rotating portion in the lower right. The associated field stretching influences dynamo action strongly.

DYNAMO ACTION

Magnetic fields with the same helical symmetry as the flow are sought. For the kinematic problem, the field $\mathbf{B}$ can be decomposed as

$$\mathbf{B} = (B\,\mathbf{H} + \mathbf{H} \wedge \nabla\chi)e^{\lambda t}$$

and the complex growth rate λ determined. Three cases are considered, when the medium outside the pipe is perfectly conducting, perfectly insulating or of the same conductivity as the fluid. Only the perfectly conducting case is presented here. The magnetic field is then zero outside of the pipe region, which has a constraining effect on the dynamo.

Pipes with no dynamo

In many cases the steady, pressure-driven pipe-flows do not in fact drive a dynamo for any R_m, except in the limit as $R_e \to \infty$. This is because the cross-pipe flow is too strong. Such a case is the square pipe $0.5 < r < 1.5$, $0 < \phi < 1$. If however, the cross-pipe flow is artifically set to zero, a kinematic dynamo occurs with a field structure which varies rapidly in the direction tangential to the pipe as shown on the right in figure 1. The asymptotic structure of the modes as $R_m \to \infty$ is analysed and agrees with the numerics.

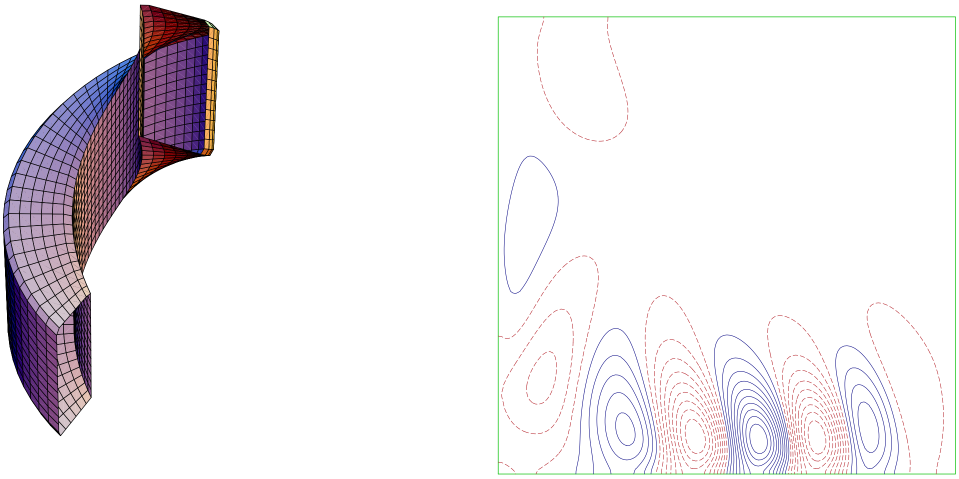

Figure 1. Left: A helical rectangular pipe for $\varepsilon = 1$, $0.5 < r < 1.5$ and $0 < \phi < \frac{2}{3}\pi$.
Right: Kinematic dynamo in a square pipe for a velocity field with $\psi = 0$ and conducting walls.

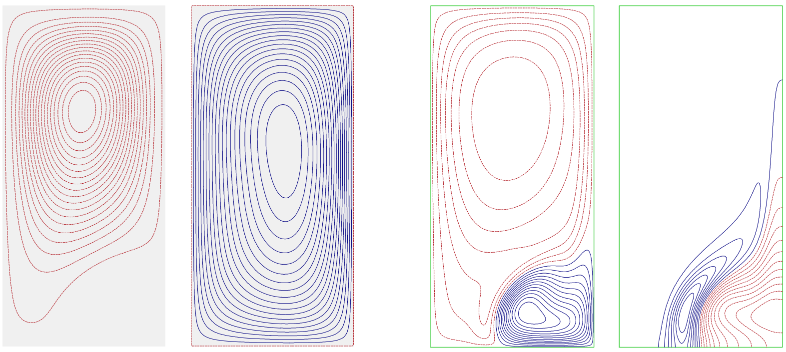

Figure 2. Dynamo for $R_m = 17^3$, $R_e = 2000$ and the pipe of figure 1. From left to right: the cross-pipe flow, ψ, and the down-pipe flow, v, the cross-pipe field, χ, and the down-pipe field, B. The inside of the pipe is on the left. The effect of field-stretching near the separation point is pronounced.

Pipes with dynamos

For the taller pipe drawn in figure 1, with $0.5 < r < 1.5$ and $0 < \phi < \frac{2}{3}\pi$, a dynamo is found for $R_e = 2 \times 10^3$ and $R_m \sim 100$. In figure 2 the growing magnetic field is shown on the right alongside the cross-pipe flow. It is clear that the stretching of the field at the separation line is an important feature of the dynamo process. The other vital ingredients are the shear in the downpipe velocity and the torsional nature of helical symmetry which drives a diffusive term involving B in the χ-part of the induction equation, a “geometrical α-effect."

Computations of the time evolution and saturation of the dynamos described here are in progress and only preliminary results are presented at the meeting. This work was supported by EPSRC grants GR/R71191/01 and GR/S87539/01 .

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