FORCE CHAIN NETWORKS AND THE STRESS RESPONSE OF GRANULAR MATERIALS

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Summary A fundamental property of any material is its response to a localized stress applied at a boundary. For granular materials consisting of hard, cohesionless particles, not even the general form of the stress response is known. Directed force chain networks (DFCNs) provide a theoretical framework for addressing this issue, and analysis of simplified DFCN models reveal both rich mathematical structure and surprising properties. For example, an exactly solvable special case, in which force chains are restricted to a discrete set of directions, exhibits a crossover from elliptic response at shallow depths to hyperbolic response at large distances. We report on recent progress toward the solution of the theory when a continuum of force chain directions is allowed.

MODELING FORCE CHAIN NETWORKS

The problem of deriving macroscopic stress equations from known microscopic or grain scale physics has proven quite difficult. An indication of just how difficult this might be can be found in almost any experiment or numerical simulation that generates images of the intensity distribution of stresses on scales of several grain diameters. In images such as that shown in Figure 1, one sees immediately that stress is concentrated on filamentary structures called force chains that support compressive stress. To the eye, these chains appear to be relatively straight on scales of up to 10 grain diameters or so and give the impression of splitting and fusing at a variety of angles. The presence of such structures indicates that passage from grain scale physics to macroscopic stresses involves solving two problems that may turn out to require completely different approaches: we must understand how grain scale physics favors the formation of chains, and also how the interactions among chains determine the macroscopic stress field. While these two tasks must ultimately be different facets of a unified theory, it may be useful to approach them separately.

Figure 1. Force chains are visible in an experimental image of plastic disks that are birefringent under strain. (Image courtesy of J. Geng and R. Behringer.)

Models of directed force chain networks (DFCNs) have recently been proposed to bridge the gap between the scale of individual chains and the macroscopic stress, leaving open the issue of how grain scale physics promotes chain formation.[1, 2] (This type of model is also referred to as a double-Y-model.[1]). In these models, the force chains are assumed to account for all of the stress. This is an idealization: the notion of a chain loses its meaning at lengths below the grain scale, and there are regions of weak stress in typical samples that appear rather homogeneous. Nevertheless, smoothly varying stresses at the grain scale could in principle be described by a high density of weak chains.

In a DFCN model, force chains are taken as primary physical entities that transmit compressive stresses only along straight line segments. The orientation of the segment determines the direction of the chain up to a sign, which is assumed to be determined by boundary conditions. Each chain is assumed to “propagate” in a straight line until either (1) it encounters a local grain configuration that cannot sustain the compressive stress associated with the chain, in which case it splits into two chains and thereby forms a vertex with one incoming and two outgoing chains, or (2) it intersects another chain, in which case it has some probability of fusing with it to form a vertex with a single outgoing chain. Material properties enter the theory through splitting and fusion functions that determine the relative probabilities of vertices with specified angles between chains.
THEORIES OF DIRECTED FORCE CHAIN NETWORKS

The structure of a DFCN is characterized at large scales by the densities of force chains propagating in each direction. There is one special case in which it has been possible to analytically determine these densities and compute the response to a localized force: the case in which the splitting and fusion functions guarantee that all vertices are three-fold symmetric, so that every force chain lies in one of six discrete directions. Examples of such vertices are shown in Figure 2. A surprising result of these calculations is that the stress response at large scales is hyperbolic; stress is transmitted along characteristic directions that do not coincide with the discrete force chain directions. For models with discrete sets of directions having $N$-fold symmetry with $N$ other than 6, solutions to the homogeneous problem are known but the response function has not yet been determined.[3] These discrete models have some features that are not readily interpreted physically. For example, though stresses remain perfectly finite, the densities of weak chains diverge.

We report on progress in obtaining homogeneous solutions for force chain networks with a continuum of force chain directions and subject to hydrostatic (isotropic) pressure. We find that the total chain density remains finite if the fusion function is chosen judiciously. We have also determined the asymptotic behavior of chain densities at large forces and small forces, which may be used to compute (or predict) quantities such as the probability distribution of contact force strengths in the underlying granular medium. The question of whether the response of these networks is elliptic or hyperbolic is currently under study.

References

