LINEAR RESPONSE OF A VISCOUS LIQUID SHEET

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Summary The paper investigates the propagation of disturbance signals originating from moving localized sources on a thin viscous liquid sheet of infinite extent. The tool of analysis is the Fourier-Laplace transformation of the linearized perturbation equations, implying an inverse Fourier-Laplace transform to reconstruct the perturbation signal. The development of the signal front from a persistently acting source is investigated in the long time limit by numerical evaluation for varicose and sinuous perturbations.

INTRODUCTION AND PROBLEM FORMULATION

Thin viscous liquid sheets are sensitive to external pressure disturbances, such as pressure fluctuations along their interfaces. In this paper the linear response of a uniform viscous liquid sheet (density ρ, kinematic viscosity ν, surface tension γ) of infinite extent to localized pressure disturbances is investigated. Fig. 1 (a) shows a sketch of the sheet with a passive

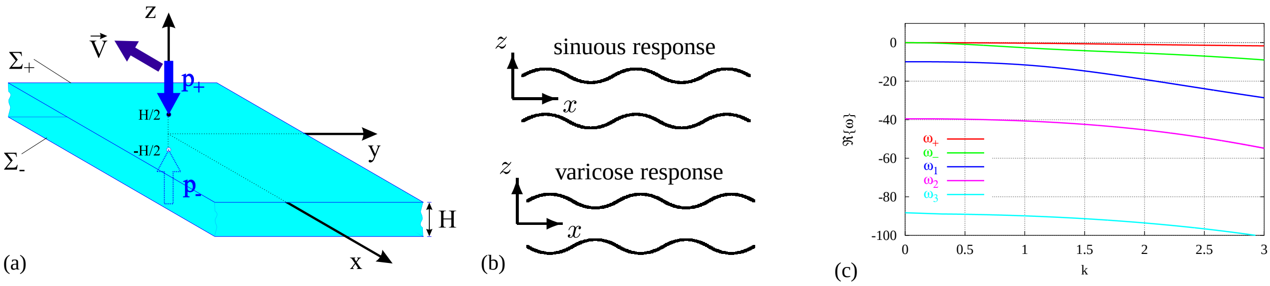


Figure 1. Sketch of a thin viscous liquid sheet (a), perturbation modes (b), varicose spectrum for $\Gamma = 1$ (c).

ambient medium adjacent to it which exerts pressure distributions $p_\pm(\vec{x}, t)$, $\vec{x} = (x, y)$, on the sheet interfaces $\Sigma_\pm$. For constant pressures $p_+ = p_- = p_0$ the sheet may be considered at rest with flat interfaces $z_\pm = \pm H/2$. Small local deviations of $p_\pm$ from p_0 induce a flow in the sheet, causing small fluctuations $\eta_\pm$ of the interfaces around its basic state: $z_\pm = \pm(H/2 + \eta_\pm)$. In Fig. 1 (a) (localized) pressure disturbances moving with constant velocity $\vec{V}$ act on the interfaces. A reflection symmetry of the basic state with respect to the plane $z = 0$ allows to consider *varicose* perturbations, that act symmetrically on the interfaces ($p_- = p_+$), and *sinuous* perturbations that act antisymmetrically ($p_- = -p_+$), separately, cf. Fig. 1 (b), with $\eta_- = \eta_+$ and $\eta_- = -\eta_+$, respectively.

RESULTS

The amplitudes η_+ of the varicose and sinuous interface deformations, $\eta_{v,s}$, are obtained by a Fourier–Laplace transform

$$\eta_{v,s}(\vec{x}, t) = -\frac{1}{2\pi} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{x}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\omega\, e^{\omega t} \frac{k\, \tilde{\mathcal{P}}(\vec{k}, \omega)}{D_{v,s}(k, \omega)}, \quad c > 0, \tag{1}$$

from the linear perturbation equations, cf. [2, 3], stated here already dimensionless by normalizing lengths by $H/2$, time by $H^2/(4\nu)$ and pressure by $4\rho\nu^2/H^2$. $\tilde{\mathcal{P}}(\vec{k}, \omega)$ denotes the Fourier–Laplace transform of the pressure disturbance $p_+(\vec{x}, t)$ and $D_{v,s}(k, \omega)$ the varicose and sinuous dispersion function, respectively:

$$D_v(k, \omega) = \Gamma^2 k^3 + (k^2 + l^2)^2 \coth(k) - 4k^3 l \coth(l), \tag{2}$$

$$D_s(k, \omega) = \Gamma^2 k^3 + (k^2 + l^2)^2 \tanh(k) - 4k^3 l \tanh(l), \tag{3}$$

with $k = |\vec{k}|$ and $l^2 = k^2 + \omega$ and with the characteristic number $\Gamma = \sqrt{\gamma H/(2\rho\nu^2)}$. The singularities of the integrand in eq. (1), the zero curves $\omega = \omega(k)$ of $D_{v,s}(k, \omega) = 0$, lie in the left half of the complex ω–plane, i.e. the response modes of the sheet are absorptive. Fig. 1 (c) illustrates the wavenumber dependence of the decay rate $\Re\{\omega\}$ for varicose mode branches, obtained numerically by parameter continuation [1]. For large times the two “soft” branches $\omega_\pm(k)$, for which $\lim_{k\to 0} \omega_\pm(k) = 0$, determine the response signal asymptotically and within the soft branches the long wave part ($k \to 0$) dominates, cf. [3].

A comparison of the decay rates of the soft branches $\omega_\pm(k)$ with their longwave approximations

$$\omega_{v,\pm}(k) \approx \pm\sqrt{4 - \Gamma^2}\, k^2 - 2k^2, \qquad \omega_{s,\pm}(k) \approx \pm i\, k - \left(\frac{2}{3} + \frac{4}{15}\Gamma^2\right) k^4 \tag{4}$$

in Fig. 2 shows for the sinuous mode that the k^4–dependence of the longwave growth rate approximates the exact solution for the soft branches only in a narrower range of k than the longwave approximation of the varicose modes.

The response (1) of the sheet to steady disturbance of the form $P_{v,s}(\vec{x}, t) = P_{0v,s}\, f_{v,s}(x + Vt, y)\theta(t)$, acting for $t > 0$, has been computed numerically in the long time limit ($t \to \infty$) for the axisymmetric form functions $f_v(x, y)$ and

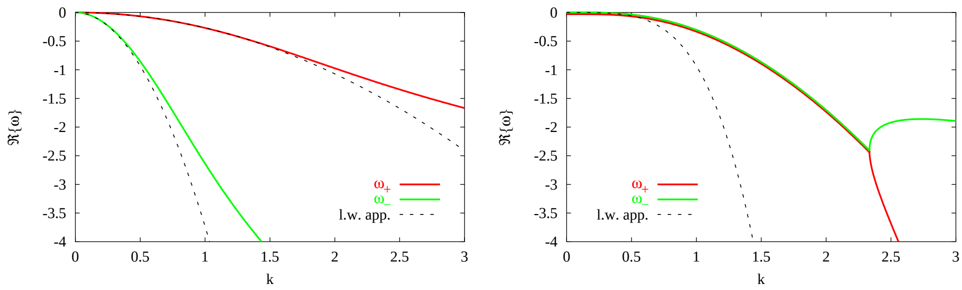
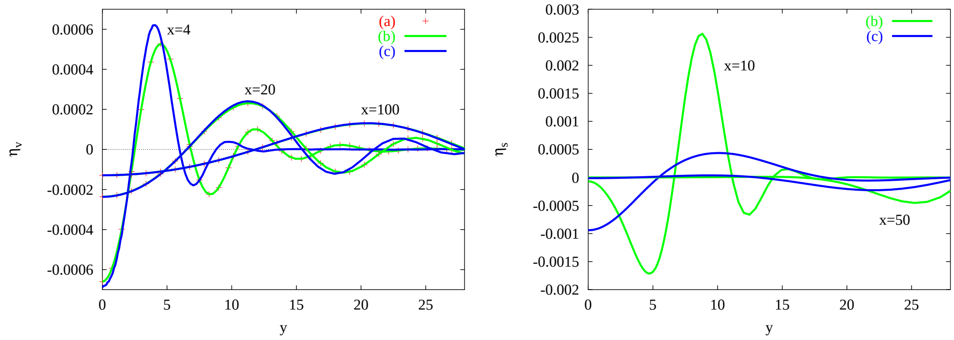


Figure 2. Solution branches of $D_{v,s}(k,\omega)$ for $\Gamma = 1$: Two soft branches $\omega_\pm(k)$ and longwave approximation (l.w. approx), for varicose modes (left) and sinuous modes (right).



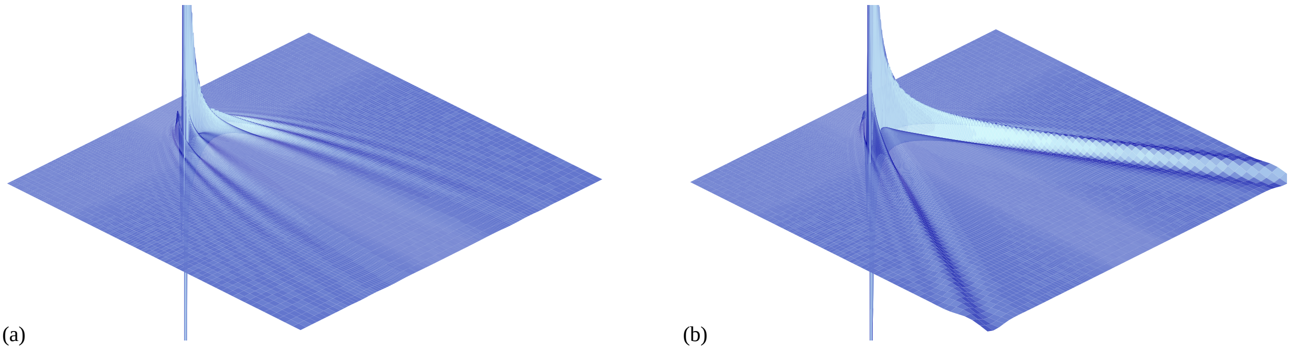
Perturbation form functions:

$$f_v(x,y) = \frac{100}{\pi}\, e^{-100\,r^2}\,,$$

$$f_s(x,y) = \left(1 - 4\,r^2\right)\, e^{-4\,r^2}\,,$$

$$r = \sqrt{x^2+y^2}\,.$$

Figure 3. Disturbance amplitude $\eta(x,y)$ along the y–direction at different distances x from the source, for $\Gamma = 10, V = 2\Gamma$ (rest frame of disturbance): varicose disturbance (left), sinuous disturbance (right). Analytical solution for longwave approximation of varicose disturbance [3] (a), numerical solutions for longwave dispersion functions (b) and for exact dispersion functions (c).



(a) (b)

Figure 4. Propagation of disturbances for the external pressure profiles $P_s(\vec{x},t)$, $\Gamma = 10, V = 2\Gamma$ (rest frame of disturbance): Numerical results for exact dispersion relations, eq. (2) for varicose disturbance (left) and sinuous disturbance (right), with $f_s(r)$ in both cases.

$f_s(x,y)$ shown in Fig. 3. The sinuous disturbance has been chosen in a way that the sheet is statically equilibrated, i.e. $\iint P_s(\vec{x},t)d\vec{x} = 0$. Fig. 3 shows the disturbance amplitudes of the sheet computed numerically, exactly with the dispersion functions, eq. (2) and (3), and in the longwave approximations [3], with eq. (4). For $\Gamma > 2$ the longwave approximation of the varicose soft modes is dispersive, so that the disturbance causes an oscillatory response, cf. Fig. 3 (left). The longwave approximation appears to describe the varicose response behaviour increasingly well with distance from the source, as expected [3]. The longwave sinuous modes are nondispersive and the response of the sheet exhibits therefore only few strongly damped undulations caused by viscous diffusion, whereas the exact evaluation takes into account also the dispersive part of the spectrum and reveals an oscillatory response, cf. Fig. 3 (right). Due to the smaller range of validity of the longwave approximation for sinuous modes, it is expected to give a less good approximation of the response than for varicose disturbances at comparable distances.

The response of the interface to varicose and sinuous disturbances, illustrated in Fig. 4, shows very good qualitative agreement with the experiments of de Luca and Costa [4].

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