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Summary We present the linear three-dimensional instabilities of the flow between exactly counter-rotating disks over the height-to-radius aspect ratio range $0.5 \leq \Gamma \leq 3$. The lowest Reynolds number threshold always corresponds to a non-axisymmetric and stationary eigenmode and the critical azimuthal wavenumber obeys $m_C \approx 2.2/\Gamma$. The axisymmetric instabilities are quite complicated, and are organized around various codimension-two points.

The von Kármán flow engendered by the differential rotation of the upper and lower bounding disks of a cylinder exhibits a large variety of phenomena, and depends on three parameters, which can be taken to be an angular velocity ratio $s \equiv \Omega_{up}/\Omega_{low}$, an aspect ratio $\Gamma \equiv H/R$, and a Reynolds number $Re \equiv \Omega_{low} R^2/\nu$. This flow is of growing interest to fluid dynamicists, but its three-dimensional patterns and transitions have as yet been explored for only a few parameter combinations, e.g. [1, 2, 3, 4, 5, 6, 7].

We focus on the case $s = -1$ in which the two disks rotate in equal and opposite directions. The system is then not only axisymmetric, but also invariant under simultaneous reflection in z and θ (or any horizontal axis), leading to the symmetry group $O(2)$. For low Re , the solution is unique and axisymmetric. For all values of Γ and Re , there exists an axisymmetric solution called the base state which is connected smoothly to the unique low- Re solution. We calculate the Reynolds numbers at which this solution becomes linearly unstable for $0.5 \leq \Gamma \leq 3.0$.

Newton's method, combined with a streamfunction-vorticity formulation, is used to calculate the base state. To calculate growth rates, we numerically integrate the Navier-Stokes equations linearized about the base state using a 3D code [6, 7] which combines a Fourier decomposition in θ with a staggered non-uniform grid in (r, z) . For an axisymmetric base state, the stability computation separates into a family of decoupled subproblems each associated with an azimuthal wavenumber m . The leading eigenvalues are extracted as half the slope of the evolution of the logarithm of the energy corresponding to each m . The thresholds $Re_m(\Gamma)$ are calculated by interpolation and shown in figure 1.

The lowest threshold $Re_C(\Gamma) \equiv \min_m Re_m(\Gamma)$ is achieved for non-axisymmetric eigenmodes over the range $0.5 \leq \Gamma \leq 3.0$ and is shown in figure 3a. The fact that these modes are stationary is a consequence of the $O(2)$ symmetry of the configuration and of the basic flow. This leads to non-axisymmetric eigenmodes which are both stationary and invariant under simultaneous reflection in z and in θ ; see figure 2. Figure 3b shows that the critical wavenumber increases approximately linearly with the radius, which is consistent with the idea that this is a generalized Kelvin-Helmholtz instability of an equatorial azimuthal shear layer occupying a constant proportion of the height.

Although the axisymmetric modes are not critical for these parameters, we present their thresholds in order to provide a complete framework which can be continued to other values of Γ or s . Figure 4a shows two pitchfork bifurcations existing for $\Gamma < 0.78$ and disappearing at an isola formation point. Figures 4b,c show two Hopf bifurcations with different frequencies co-existing for $\Gamma < 0.55$, where one of them disappears at a Takens-Bogdanov point. The rapid variation of the axisymmetric Hopf bifurcation threshold seen in figure 1 near $\Gamma = 1.6$ is due to an intersection of two complex eigenvalues. All the codimension-two points, both non-axisymmetric and axisymmetric, are shown in the table.

Competing wavenumbers or bifurcations	Γ	Re	Description
(4,3)	0.63	365	
(3,2)	0.95	310	
(2,1)	1.64	330	
H_1/P_2	0.51	2802	Takens-Bogdanov point isola formation point intersection of two complex eigenvalues
H_1/H_2	0.535	2602	
H_2/P_2	0.55	2309	
P_1/P_2	0.78	700	
H_1	1.59	1860	

References

- [1] Y.A. Gelfgat, P.Z. Bar-Yoseph, A. Solan: Three-dimensional instability of axisymmetric flow in a rotating lid-cylinder enclosure, *J. Fluid Mech.* **438**, 363, 2001.
- [2] J.M. Lopez, J.E. Hart, F. Marques, S. Kittelman, J. Shen: Instability and mode interactions in a differentially-driven rotating cylinder, *J. Fluid Mech.* **462**, 383, 2002.
- [3] L. Schouveiler, P. Le Gal, M.-P. Chauve: Instabilities of the flow between a rotating and a stationary disk, *J. Fluid Mech.* **443**, 329, 2001.
- [4] G. Gauthier, P. Gondret, F. Moisy, M. Rabaud.: Instabilities of the flow between co and counter-rotating disks, *J. Fluid Mech.* **473**, 1, 2002.
- [5] E. Serre, E. Crespo del Arco, P. Bontoux: Annular and spiral patterns in flows between rotating and stationary disks, *J. Fluid Mech.* **434**, 65, 2001.
- [6] C. Nore, L.S. Tuckerman, O. Daube, S. Xin: The 1 : 2 mode interaction in exactly counter-rotating von Kármán swirling flow, *J. Fluid Mech.* **477**, 51, 2003.
- [7] C. Nore, M. Tartar, O. Daube, L. S. Tuckerman: Survey of instability thresholds of flow between exactly counter-rotating disks, *J. Fluid Mech.*, to appear.

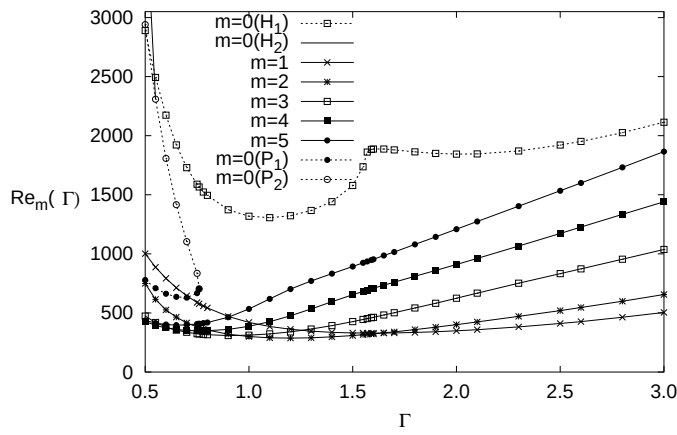
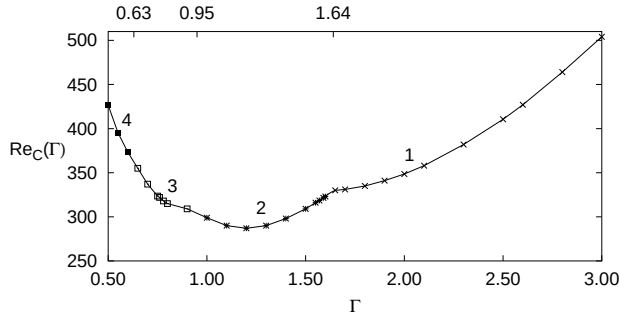


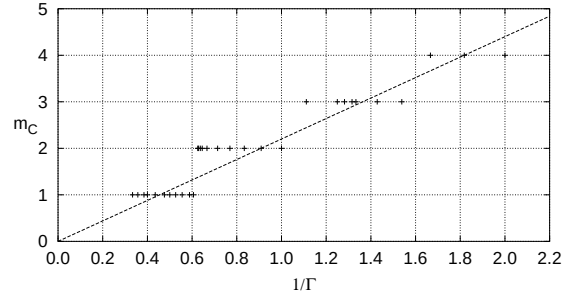
Figure 1. Thresholds $Re_m(\Gamma)$ for azimuthal modes $m = 0$ to 5 as functions of the aspect ratio Γ . The modes $m = 1$ to 5 are stationary while the $m = 0$ mode is stationary for $m = 0$ (P_1) and (P_2) and oscillatory for $m = 0$ (H_1) and (H_2).



Figure 2. Vertical velocity contours of an $m = 3$ eigenvector at $z = \Gamma/4$, $z = 0$, and $z = -\Gamma/4$.

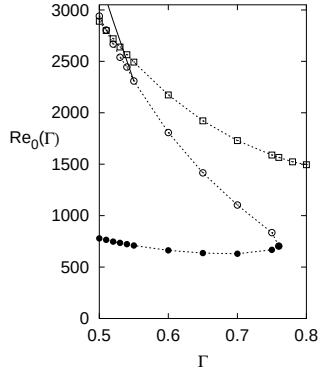


(a)

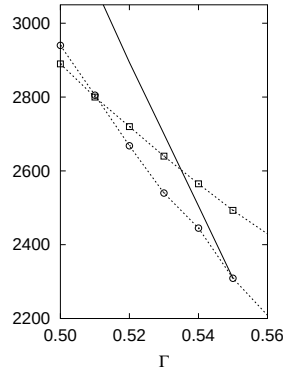


(b)

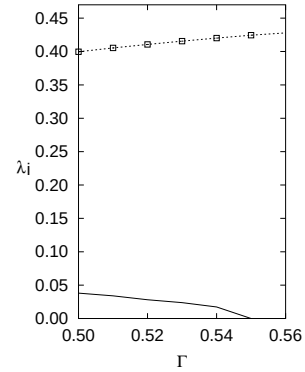
Figure 3. (a) Critical Reynolds number $Re_C(\Gamma)$ as a function of Γ realised for different critical wavenumbers: $m = 1$ ($\times$), $m = 2$ ($*$), $m = 3$ ($\square$), $m = 4$ ($\blacksquare$). (b) Critical wavenumber m_C ($+$) as a function of $1/\Gamma$ with the indicative dashed line $m = 2.2/\Gamma$.



(a)



(b)



(c)

Figure 4. (a) Axisymmetric thresholds: two stationary bifurcations P_1 ($\bullet$) and P_2 ($\circ$) and two Hopf bifurcations H_1 ($\square$) and H_2 ($—$). P_1 and P_2 disappear at the isolation formation point at $\Gamma = 0.78$. (b) Enlargement of (a) showing the disappearance of H_2 at the Takens-Bogdanov point at $\Gamma = 0.55$. (c) Frequencies for H_1 ($\square$) and H_2 ($—$).