# STABILITY OF LAGRANGIAN IDEAL FLOWS

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Summary: A general method is presented to investigate the stability of ideal incompressible flows described in Lagrangian representation. Based on the theory of short-wavelength instabilities, the problem is reduced to a transport equation which involves only the distortion matrix of the equilibrium flow. Theory is applied to Gerstner's rotational free-surface gravity waves. It is shown that they are three-dimensionally unstable when their steepness exceeds $1/3$.

## INTRODUCTION

The motion of a continuous medium may be described either by the trajectories of its material particles or by the velocity field expressed at any geometric point of space. Although both have been formalized by Euler himself [1], these alternative but equivalent descriptions are called respectively Lagrangian and Eulerian. Both present their own advantages. The Lagrangian description is more convenient to describe the deformations of the medium and is generally used in elasticity theory. Hydrodynamics clearly prefers the Eulerian approach probably because of the extreme complexity of the viscous terms. In ideal flows however, some explicit solutions are known in Lagrangian representation, such as Gerstner's waves discovered in 1802 [1] or unsteady vortices discovered recently [2]. However, it seems that the stability of Lagrangian flows has never been addressed in the past. We propose here a general method based on the theory of short-wave instabilities [3] and apply it to Gerstner's waves [4].

## KINEMATICS AND LAGRANGIAN DYNAMICS

Let $\boldsymbol{U}(\boldsymbol{x},t)$ be the Eulerian velocity field of a continuous medium, and $\boldsymbol{X}(t;\boldsymbol{a})$ the trajectories of the material particles, where $\boldsymbol{a}$ is the Lagrangian label. We remind that $\boldsymbol{a}$ is not necessary the initial position of the particle. We assume that the motion is incompressible and smooth. The same notation will be used for any scalar, vector or tensor field $f$ described in either Eulerian or Lagrangian representation: $f(\boldsymbol{X}(t;\boldsymbol{a}),t) \equiv f(t;\boldsymbol{a}) \equiv f(t)$. Material derivative will be noted $\dot{f}$ and initial data $f_0$. If $\dot{f}=0$, $f$ is a Lagrangian invariant. Trajectories are solutions of $\dot{\boldsymbol{X}} = \boldsymbol{U}$.
Deformations of the medium are characterized by the matrix $\boldsymbol{\mathcal{F}}$ with components [1]:

$$\mathcal{F}_{ij} = \partial X_i/\partial a_j. \tag{1}$$

Continuity and incompressibility assumptions require that $\det \boldsymbol{\mathcal{F}}(t) = \det \boldsymbol{\mathcal{F}}_0 \neq 0$. Therefore $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{F}}^{-1}$ is defined. $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$ satisfy respectively:

$$\dot{\boldsymbol{\mathcal{F}}} = \boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{F}}, \quad \dot{\boldsymbol{\mathcal{G}}} = -\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}, \tag{2a, b}$$

where $\boldsymbol{\mathcal{L}}$ is the velocity gradient tensor with components $\mathcal{L}_{ij} = \partial U_i/\partial x_j$.
The following kinematical relation may be established from (2):

$$\dot{\boldsymbol{\mathcal{F}}}^T\boldsymbol{\mathcal{F}} - \boldsymbol{\mathcal{F}}^T\dot{\boldsymbol{\mathcal{F}}} = -2\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{F}}, \tag{3}$$

where $\boldsymbol{\mathcal{A}} = \frac{1}{2}(\boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}}^T)$ is the skew-symmetric vorticity tensor. In an ideal incompressible flow, it may be shown that the right hand side of (3) is a Lagrangian invariant [2]. The motion of the fluid may be fully described by the evolution of the matrix $\boldsymbol{\mathcal{F}}(t)$, solution of (3) plus incompressibility $\det \boldsymbol{\mathcal{F}} = \det \boldsymbol{\mathcal{F}}_0$ and consistency relations. Exact solutions have been discovered with this formulation, such as precessing or curvilinear vortices [2].

## THE THEORY OF SHORT-WAVE INSTABILITIES

The linear stability of an Eulerian equilibrium flow with velocity $\boldsymbol{U}$ is characterized by the growth or decay of infinitesimal disturbances $\boldsymbol{u}$ governed by the linearized Euler equations. The theory of short-wave instabilities consists in considering the evolution of a rapidly-varying WKB wave packet $\boldsymbol{u} = \boldsymbol{v}\mathrm{e}^{\mathrm{i}\phi/\varepsilon}$ with $\varepsilon \ll 1$. By defining the wave vector $\boldsymbol{\xi} = \boldsymbol{\nabla}\phi$, it may be shown that the stability problem is reduced to a system of ordinary differential equations that evolves along the trajectories of the equilibrium flow [3]:

$$\dot{\boldsymbol{X}} = \boldsymbol{U}, \quad \dot{\boldsymbol{\xi}} = -\boldsymbol{\mathcal{L}}^T\boldsymbol{\xi}, \quad \dot{\boldsymbol{v}} = (2\boldsymbol{\xi}\otimes\boldsymbol{\xi}/|\boldsymbol{\xi}|^2 - \boldsymbol{\mathcal{I}})\boldsymbol{\mathcal{L}}\boldsymbol{v}. \tag{4a, b, c}$$

Here $\boldsymbol{\mathcal{L}}$ is the basic velocity gradient tensor, $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$ the tensor with components $\xi_i\xi_j$, and $\boldsymbol{\mathcal{I}}$ the identity matrix. This system is completed by initial data for the wave vector and the velocity amplitude: $|\boldsymbol{\xi}_0| = |\boldsymbol{v}_0| = 1$ and $\boldsymbol{\xi}_0 \perp \boldsymbol{v}_0$. It is proved that *the equilibrium flow (steady or not) is unstable if there exists at least one trajectory along which* $|\boldsymbol{v}(t)|$ *grows unboundedly* [3].
A general solution of (4) is not known, but in some circumstances this set of equations may be locally integrated. Various classical stability results have been recovered and generalized in incompressible or compressible ideal flows (Rayleigh

and Leibovich-Stewartson criteria, hyperbolic and elliptical instabilities), and new instability mechanisms have been discovered in steady or unsteady flows. A review may be found in [3].

The theory of short-wave instabilities is greatly simplified when the equilibrium flow is described in Lagrangian form [4]. Indeed the trajectories being known explicitly, $(4a)$ is already solved. The distortion matrix $\boldsymbol{\mathcal{F}}$ defined in (1), its inverse $\boldsymbol{\mathcal{G}}$, their time derivative and their initial value may then be computed by differentiation. Also, it may be shown from (2) that the wave vector $\boldsymbol{\xi}(t)$ solution of $(4b)$ with initial data $\boldsymbol{\xi}_0$ admits the following explicit expression: $\boldsymbol{\xi}(t) = \boldsymbol{\mathcal{G}}^T(t)\boldsymbol{\mathcal{F}}_0^T\boldsymbol{\xi}_0$. Furthermore the basic velocity gradient $\boldsymbol{\mathcal{L}}$ involved in the amplitude equation $(4c)$ may also be calculated thanks to the kinematical relations (2). As a consequence, instead of system (4), the local stability of a Lagrangian ideal flow is reduced to the study of the following transport equation:

$$\dot{\boldsymbol{v}} = \left(\boldsymbol{\mathcal{F}} - 2\boldsymbol{\mathcal{G}}^T \frac{\boldsymbol{n}_0 \otimes \boldsymbol{n}_0}{\boldsymbol{n}_0^T \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^T \boldsymbol{n}_0}\right) \dot{\boldsymbol{\mathcal{G}}}\boldsymbol{v}, \quad \boldsymbol{n}_0 = \boldsymbol{\mathcal{F}}_0^T \boldsymbol{\xi}_0, \tag{5}$$

with initial data verifying $|\boldsymbol{\xi}_0| = |\boldsymbol{v}_0| = 1$ and $\boldsymbol{\xi}_0 \perp \boldsymbol{v}_0$.

## APPLICATION TO GERSTNER'S WAVES

Theory is now applied to Gerstner's waves, an exact solution of Euler equations. In the Galilean frame $(O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, let $\rho$ be the density of an incompressible inviscid free-surface flow, subjected to gravity $-g\boldsymbol{j}$. The particle paths in Gerstner's waves are given parametrically by [1]:

$$X(t) = a + k^{-1}\mathrm{e}^{kb}\sin(ka - \omega t), \quad Y(t) = b - k^{-1}\mathrm{e}^{kb}\cos(ka - \omega t), \quad Z(t) = c, \tag{6a, b, c}$$

where $k > 0$ is the spatial wavenumber, $\omega = \sqrt{gk}$ the frequency, and $\boldsymbol{a} = (a, b, c)$ with $b < 0$ is the Lagrangian label. Trajectories are circles of radius $\mathrm{e}^{kb}/k$ and center $(a, b)$ in planes $Z =$ const.

Pressure field reads as $P = -\rho g b + \frac{1}{2}\rho\omega^2\mathrm{e}^{2kb}/k^2 +$ const. Surfaces of constant pressure are parametrized by $b$ and are represented on figure 1. They are trochoids with crest-trough amplitude $2\mathrm{e}^{kb}/k$. Any pressure level, say $P(b_0)$, may be chosen as free surface for the flow. The steepness parameter of the free-surface profile thus reads $\mathrm{e}^{kb_0}$ (half amplitude multiplied by wave number). When time varies those gravity waves propagate from left to right with celerity $\omega/k$. Contrary to Stokes waves, Gerstner's waves are rotational with vorticity maximum and the free surface and decreasing rapidly with depth. Therefore they cannot be created from a state of rest nor by pressure forces.

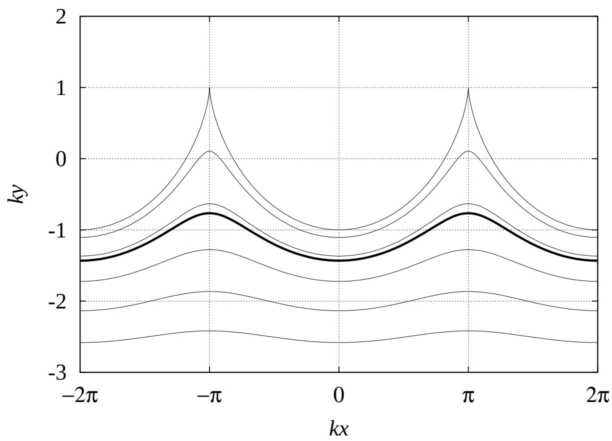


**Figure 1.** Contour lines of pressure of Gerstner's waves at $t = 0$, plotted for $kb = -2.5$ (lower curve) to $kb = 0$ (upper curve) with increment $0.5$. Any level of constant pressure may be considered as the free surface for the flow. The thick line corresponds to $kb = -\ln 3$ above which Gerstner's waves are unstable to short-wavelength disturbances.

The equilibrium flow being two-dimensional in the plane $(O; \boldsymbol{i}, \boldsymbol{j})$, local disturbances characterized by $\boldsymbol{\xi}_0 = \boldsymbol{k}$ lead to the simplified equation $\dot{\boldsymbol{v}} = \boldsymbol{\mathcal{F}}\dot{\boldsymbol{\mathcal{G}}}\boldsymbol{v}$ instead of (5). Both tensors $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$ may be computed explicitly from (6) and the solution may be obtained in closed form [4]. The main result is that *Gerstner's waves are unstable when the free-surface vorticity exceeds* $\omega/4$, *or equivalently when the steepness parameter exceeds* $1/3$. The unstable region is represented on figure 1.

### References


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