# A FRACTAL COHESIVE CRACK MODEL

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## 1. MATHEMATICAL PRELIMINARIES

In order to investigate the size effects associated with imperfect materials weakened by voids and crack-like defects, we need to establish a characteristic length defined by the material microstructure and geometry of the defect. A certain length parameter, R, is embedded in all cohesive crack models. Equilibrium between the restraining tractions $S(X) = S_0G(X)$ applied over the end zone itself, is determined by the finiteness condition

$$K_{tot}(\sigma, S) = K_I(\sigma) + K_{coh}(S) = 0 \tag{1.1}$$

With the pressure σ applied to the crack surface, $|X| \le a$, and the pressure p = σ – S(X) applied over the end zones, $a \le |X| \le a + R$, equation (1.1) reads

$$2\sqrt{\frac{a+R}{\pi}}\left\{\int_0^a \frac{\sigma\, dX}{\sqrt{a^2 - X^2}} + \int_a^{a+R} \frac{(\sigma - S(X))dX}{\sqrt{a^2 - X^2}}\right\} = 2\sqrt{\frac{a+R}{\pi}}\left\{\frac{\pi}{2}\sigma + \int_a^{a+R} \frac{-S(X)\, dX}{\sqrt{a^2 - X^2}}\right\} = 0 \tag{1.2}$$

When the nondimensional loading parameter $\pi\sigma/2S_0$ is denoted by Q, the condition (1.2) can be re-written as

$$Q = \int_a^{a+R} \frac{\widetilde{G}(X)\, dX}{\sqrt{a^2 - X^2}} \tag{1.3}$$

Here, the cohesive stresses are given by the Wnuk-Legat distribution law, cf. [2],

$$\widetilde{G}(X) = \left(\frac{X-a}{R}\right)^n \exp\left[\omega\left(\frac{a-X}{R}\right)\right] \tag{1.4}$$

The material parameters n and ω are subject to experimental determination at the mesomechanical level. If the pertinent variables are nondimensionalized as follows,

$$\frac{a}{a+R} = m,\ \lambda = \frac{x}{R} = \frac{X-a}{R} = \frac{x-m}{1-m},\ x = \frac{X}{a+R} \tag{1.5}$$

Equation (1.5) reads

$$Q = \int_0^1 \frac{G(\lambda)(1-m)\,d\lambda}{\sqrt{1-\left[(1-m)\lambda + m\right]^2}} \tag{1.6}$$

In what follows we shall restrict the considerations to the case of R<<a, or $m \to 1$. This restriction is pertinent to the "small scale yielding" (ssy) condition frequently met in analyses of fracture in quasi-brittle solids. For this limiting case, when we consider (1-m) as a small quantity, the integral in Equation (1.6) can be simplified as follows

$$Q \cong \sqrt{\frac{1-m}{2}} \int_0^1 \frac{G(\lambda)}{\sqrt{1-\lambda}}\, d\lambda \tag{1.7}$$

Replacing (1-m) by R/a, we obtain

$$Q \cong \sqrt{\frac{R}{2a}} \int_0^1 \frac{G(\lambda)}{\sqrt{1-\lambda}}\, d\lambda \tag{1.8}$$

in which G(λ) results from G(X), when the appropriate transformation of variables, cf. Eq. (1.5), is completed. It reads

$$G(\lambda, n, \omega) = \lambda^n \exp\left[\alpha(1-\lambda)\right] \tag{1.9}$$

Multiplication of Q in (1.8) by the factor $2S_0\sqrt{a/\pi}$ , where $S_0$ denotes the local value of the yield stress measured at the crack front, converts this expression to the so-called "cohesion modulus" (used in somewhat different form by Barenblatt [1]), namely

$$K_{coh} = \sqrt{\frac{2R}{\pi}} S_0 \int_0^1 \frac{G(\lambda, n, \omega)}{\sqrt{1-\lambda}}\, d\lambda \tag{1.10}$$

With the integral contained in (1.10) denoted by $W_0(n,\omega)$, it is customary to solve Equation (1.10) for the characteristic length

$$R = \frac{\pi}{2W_0^2} \left( \frac{K_{coh}}{S_0} \right)^2 \tag{1.11}$$

This value is considered to represent the characteristic length parameter reflecting the microstructural and mesomechanical properties of the material. The equations provided in this section are valid for a smooth crack.

## 2. A FRACTAL MODEL OF THE COHESIVE CRACK AND THE CHARACTERISTIC LENGTH

Recently proposed fractal model of Wnuk and Yavari [3] suggests a mathematical simplification based on associating a smooth crack to a stress field generated around a fractal crack. This is the so-called method of imaginary smooth crack. The approach provides a useful approximation to the problem at hand, including the cohesive aspects of a fractal crack. In the solutions now obtained a new variable enters: the fractal dimension of the crack. The fractal dimension, D, usually a non-integer, is a measure of how strongly a given entity diverges from its Euclidean counterpart. As a geometrical characteristic of the fracture surface, D enters as a new variable in most pertinent equations of the fractal fracture mechanics. Let us mention just one such relation – for a fractal version of the Griffith crack the familiar singularity of $r^{-1/2}$ is replaced by a somewhat weaker singularity for the near-tip stress, $r^{-\alpha}$, where for a self-similar crack α depends on the fractal dimension D, namely (see Yavari, et al. [4] and Yavari [5])

$$\alpha = \frac{2-D}{2}, \qquad 1 \le D \le 2 \tag{2.1}$$

The model allows one to generalize the formula for the cohesive modulus of a smooth crack (1.10), to the one representing a fractal crack, namely

$$K_{coh}^{f} = \frac{2}{\pi} S_0 \left( \frac{\pi R_f}{2} \right)^{\alpha} \int_0^1 \frac{G(\lambda, n, \omega)}{(1-\lambda)^{\alpha}} d\lambda \tag{2.2}$$

The inverse relation that relates the characteristic length R to the material properties such as $S_0$, n, ω, $K^f_{coh}$, and the fractal geometry represented by the order of singularity α, is given now as

$$R_f = \left( \frac{\pi}{2} \right)^{\frac{1-\alpha}{\alpha}} \left[ K_{coh}^{f} / S_0 \right]^{\frac{1}{\alpha}} \left\{ \int_0^1 \frac{G(\lambda, n, \omega)}{(1-\lambda)^{\alpha}} \; d\lambda \right\}^{-\frac{1}{\alpha}} \tag{2.3}$$

Index and superscript "f" have been added to designate quantities pertinent to a fractal crack. It is readily seen that for the limiting case of α = ½, the formulae (1.10) and (1.11), valid for a smooth crack, are recovered. With the notation

$$W(\alpha, n, \omega) = \int_0^1 \frac{G(\lambda, n, \omega)}{(1-\lambda)^{\alpha}} d\lambda, \qquad R_c^f = \left( K_{coh}^{f} / S_0 \right)^{\frac{1}{\alpha}} \tag{2.4}$$

equation (2.3) can be re-written in a nondimensional form

$$\frac{R_f}{R_c^f} = \frac{(\pi/2)^{\frac{1-\alpha}{\alpha}}}{W^{\frac{1}{\alpha}}(\alpha,n,\omega)} \tag{2.5}$$

Similarly, for a smooth crack, see Eqs, (1.10) and (1.11), we have

$$\frac{R}{R_c} = \frac{\pi}{2W_0^2}\ , \quad R_c = (K_{coh}/S_0)^2 \tag{2.6}$$

Dividing equations (2.5) and (2.6) side by side, one obtains a measure of the material characteristic length associated with a fractal crack

$$\Theta(\alpha) = \frac{R_f R_c}{RR_c^f} = \left(\frac{\pi}{2}\right)^{\frac{1-2\alpha}{\alpha}} \frac{W_0^2}{W^{\frac{1}{\alpha}}} \tag{2.7}$$

It is easy to verify that for the limiting case of α = ½, when W $\to$ $W_0$, the function Θ(α) reduces to one, as expected. Fig. 1a shows the dependence of Θ on the fractal singularity exponent α, plotted for the range $\frac{1}{4} \le \alpha \le \frac{1}{2}$, while Fig. 1b shows a similar function, for which the fractal dimension D was chosen as an independent variable rather than α. As can be seen, the range ($\frac{1}{4}, \frac{1}{2}$) for α corresponds to the range (1, $\frac{3}{2}$) for D.

There is a problem with physical interpretation of a fractal crack behavior when D approaches 2. In this limiting case the crack resembles a 2D object spread over a plane (a plane filling curve). For this case, the stress intensity factor, cf. Wnuk and Yavari [3]

$$K_I^f = \frac{\sigma\sqrt{\pi a^{2\alpha}}}{\pi^{2\alpha}} \int_0^1 \frac{(1+s)^{2\alpha} + (1-s)^{2\alpha}}{(1-s^2)^{\alpha}} ds \tag{2.8}$$

attains the value $2\sqrt{\pi}\sigma$. Note that the crack length "a" is suppressed entirely, and the entity $2\sqrt{\pi}$ should thus be interpreted as Nueber's stress magnification factor (rather than a stress intensity factor) corresponding to a certain 2D void. Assuming the void to be in the shape of a blunted crack with a finite root radius ρ, and the crack front identified at r = ρ/2, cf. Wnuk and Kriz [6], the stress at the crack front may be evaluated as follows

$$\sigma_{\max} = \frac{2K_I}{(\pi\rho)^{\frac{1}{2}}} = \frac{2\sigma\sqrt{\pi a}}{(\pi\rho)^{\frac{1}{2}}} \tag{2.9}$$

Setting it equal to the value predicted by the fractal model

$$\sigma^{f}_{\max} = \left[ \frac{\chi(\alpha)\sigma\sqrt{\pi\, a^{2\alpha}}}{(2\pi\, r)^{\alpha}} \right]_{r=\rho/2} = \frac{\chi(\alpha)\sigma\sqrt{\pi}\, a^{\alpha}}{(\pi\,\rho)^{\alpha}} \tag{2.10}$$

leads to an expression defining the root radius of the fractal crack when $D \to 2$, or $\alpha \to 0$, namely

$$\frac{\chi(\alpha)\, a^{\alpha-\frac{1}{2}}}{(\pi\,\rho)^{\alpha}} = \frac{2}{(\pi\,\rho)^{\frac{1}{2}}} \tag{2.11}$$

## 3. CONCLUSIONS

It has been demonstrated that a solid weakened by a fractal crack possesses a characteristic length, which is determined by the mesomechanical properties of the material, such as $S_0$, n, ω, $K^{f}_{coh}$ and also by the fractal dimension of the crack, D. Two measures of this length have been suggested here for two intervals of the fractal dimension D. For the fractal dimension ranging from 1 (a smooth crack limit) to 1.846, the characteristic length $R^{f}_{c}$ is used, and it varies between 1 for D = 1, and 201.8 for D = 1.846. It should be noted that the value of 201.8 is three to four orders of magnitude greater than the values of the characteristic length observed in ductile metals. This result is of significance when the size effects related to fracture in cementitious materials is interpreted. The value of D = 1.846 is used as a cut-off value for a fractal dimension. Beyond this limit a root radius of a hypothetical blunted crack (equivalent to a fractal when D approaches 2) is suggested as a measure of the characteristic material length. In the limit of D = 2, this root radius equals a/π, where "a" denotes the nominal crack length.

## REFERENCES


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