GENERALIZATION OF THE ESHELBY METHOD FOR SOLVING ELASTICITY PROBLEMS WITH PHASE TRANSFORMATIONS AND FOR PIECEWISE HOMOGENEOUS BODIES

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Consider an elastic problem of phase transformation in an inclusion G, located in an infinite isotropic elastic solid. Let us suppose that the phase transformation can be described by the eigenstrains $e^p_{ij,j}(x)$ $(e^p_{ij}(x)=0,\ x\notin G)$. We denote the elastic constants by λ_M and μ_M. The total strains can be written as

$$e_{ij}(x)=e^F_{ij}(x)+e^P_{ij}(x) \tag{1}$$

where $e^F_{ij}(x)$ are elastic strains.

Let us introduce the tensors

$$\sigma^d_{ij}(x)=\lambda_M\theta(x)\delta_{ij}+2\mu_M e_{ij}(x)\,,\quad \sigma^p_{ij}(x)=\lambda_M\theta^p(x)\delta_{ij}+2\mu_M e^p_{ij}(x) \tag{2}$$

Here $\theta(x)=\sum_{k=1}^{3}e_{kk}(x),\quad \theta^P(x)=\sum_{k=1}^{3}e^P_{kk}(x)$.

It is shown that the tensor $\sigma^d_{ij}(x)$ is calculated by the formulae

$$\sigma^d_{ij}(x)=-\int_G\frac{\partial T^k_{ij}(x-y)}{\partial x_m}\sigma^P_{km}(y)dy \tag{3}$$

where $T^k_{ij}(x)$ is the stress tensor related to the concentrated loads, $T^k_{ij,j}(x)+\delta_{ik}\delta(x)=0$, summation by the repeated indices is supposed here and later on.

The tensor of actual stresses $\sigma_{ij}(x)$ is expressed by means of the tensors $\sigma^d_{ij}(x)$ and $\sigma^p_{ij}(x)$ as follows $\sigma_{ij}(x)=\sigma^d_{ij}(x)-\sigma^P_{ij}(x)$.

If $e^p_{ij}(x)=\mathrm{const},\ x\in G$, and as a consequence $\sigma^p_{ij}(x)=\mathrm{const},\ x\in G$, then the problem is reduced to the case considered by Eshelby, and formulae (3) are reduced to the formulae, obtained by Eshelby

$$\sigma^d_{ij}(x)=\int_{\partial G}T^k_{ij}(x-y')t^P_k(y')dS_{y'},\quad t^P_k(y')=\sigma^p_{km}n_m(y') \tag{4}$$

Here $n(y')=(n_1(y'),n_2(y'),n_3(y'))$ is a unit outward normal to ∂G at $y'\in\partial G$ and $\sigma^p_{km}(x)=\sigma^p_{km}=\mathrm{const},\ x\in G$.

It is interesting to note that when the domain G tends to two-dimensional surface the actual stresses tend to zero outside G. This fact probably denotes that the Eshelby model is not appropriate for the description of the process of second phase growth.

When G is an elliptical cylinder the explicit analytical expressions are obtained for $\sigma_{ij}(x)$ both inside and outside G. This problem was solved by Eshelby but the explicit expressions for stresses outside G were not obtained.

At the next step we apply the equivalent inclusion method for solving the inhomogeneous problem for the inclusion G of arbitrary shape. Let us suppose that elastic constants of the inclusion are λ_I and μ_I and at the infinity constant deformations e^A_{ij} are applied. Using the generalized Eshelby method we reduced the problem to the following system of integral equations inside the inclusion

$$
\begin{aligned}
&\frac{\mu_M}{\mu_I}\sigma_{ij}^{0}(x)+\frac{\mu_M(\nu_M-\nu_I)}{\mu_I(1+\nu_I)(1-2\nu_M)}\Theta^{0}(x)\delta_{ij}= \\
&=-\int_G \frac{\partial T_{ij}^{k}(x-y)}{\partial x_m}\left(\frac{\mu_M}{\mu_I}\sigma_{km}^{0}(y)+\frac{\mu_M(\nu_M-\nu_I)}{\mu_I(1+\nu_I)(1-2\nu_M)}\Theta^{0}(y)\delta_{km}-\sigma_{km}^{0}(y)\right)dy+ \\
&+\int_{\partial G} T_{ij}^{k}(x-y')\left(t_k^{MA}(y')-t_k^{IA}(y')\right)dS_{y'},\quad x\in G
\end{aligned}
\tag{5}
$$

Here $T_{ij}^{k}(x)$ is the stress tensor for concentrated loads in a homogeneous solid with elastic moduli λ_M and μ_M, $\sigma_{ij}^{IA}=\lambda_I\theta^A\delta_{ij}+2\mu_I e_{ij}^{A}$, $\sigma_{ij}^{MA}=\lambda_M\theta^A\delta_{ij}+2\mu_M e_{ij}^{A}$, $\Theta^A=\sum_{k=1}^{3}e_{kk}^{A}$,

$t_k^{IA}(y')=\sigma_{km}^{IA}n_m(y')$, $t_k^{MA}(y')=\sigma_{km}^{MA}n_m(y')$, $\sigma_{ij}^{A}(x)=\begin{cases}\sigma_{ij}^{IA}, x\in G\\ \sigma_{ij}^{MA}, x\notin G\end{cases}$, $\sigma_{ij}(x)=\sigma_{ij}^{A}(x)+\sigma_{ij}^{0}(x)$, $\sigma_{ij}(x)$ is the actual stress tensor, $\theta^{0}(x)=\sum_{k=1}^{3}\sigma_{kk}^{0}(x)$, ν_I and ν_M are Poisson's ratios for the inclusion and matrix, respectively.

When G is an elliptical cylinder the explicit analytical expressions are obtained for the actual stress tensor $\sigma_{ij}(x)$. The problem for the elliptical inclusion was solved about 50 years ago but the explicit expressions for the stresses outside the inclusion were not presented.

The problem of phase transformation of the domain G which can be described by the eigenstrains $e_{ij}^{p}(x)$ and the change of elastic moduli from λ_M, μ_M to λ_I, μ_I is considered too. A system of integral equations relatively components of the stress tensor inside the inclusion, which is similar to the system (5) is constructed. When the inclusion is an elliptical cylinder the explicit analytical expressions for the stress tensor both inside and outside the inclusion are obtained for this problem also.