EXACT NONLINEAR THEORY OF BENDING AND TORSION OF ELASTIC RODS

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Summary The Saint-Venant's problems for prismatic rod, rod as sector of a circular ring and rod in the shape of a helical spring are considered on the basis of exact three-dimensional equations of nonlinear elasticity. The central point of the investigation is the semi-inverse solution of three-dimensional equations of a non-linearly elastic body statics. By means of semi-inverse solutions, spatial problems of nonlinear elastostatics are reduced to two-dimensional problems.

On the basis of exact three–dimensional equations of nonlinear elasticity we consider the Saint–Venant's problems for bodies of different shapes, i.e. problems of large tension, bending and torsion deformations of prismatic and curved beams loaded with end forces and moments. The central point of our investigation is the semi–inverse solution of three–dimensional equations of a non–linearly elastic body statics [1]. The solutions found represent two–parameteric sets of £nite deformations de£ned by Cartesian, cylindrical or special curvilinear coordinates. At these deformations the initial three–dimensional system of nonlinear equations of elastic medium equilibrium is reduced to a system of equations with two independent variables.

The semi–inverse solutions mentioned describe the following types of large deformations: torsion and tension–compression of a prismatic beam [2], spatial bending of a prismatic body, spatial bending of sector of a circular ring, tension and torsion of a curved rod in the shape of a helical spring [3], bending of a helical spring, tension and torsion of a naturally twisted prismatic body, bending of a rectilinear naturally twisted rod. The special cases of these classes of deformations are pure bending of a prismatic body [4], straightening of sector of a circular ring, pure bending of a curved beam. Due to the use of semi–inverse solutions, spatial problems of nonlinear elastostatics listed above are reduced to two–dimensional boundary value problems for ¤at domain in the shape of cross–section of a prismatic or curved beam.

Solution of two–dimensional boundary value problems obtained allows to exactly satisfy the equilibrium equations in body volume and boundary conditions at rod lateral surface, which is supposed free from load. The boundary conditions at the ends of rod are ful£lled approximately, in Saint–Venant's integral sense.

Let's cite semi–inverse solutions circumscribing tension–compression and torsion of a prismatic beam and spatial bending of a cylindrical body as an example. Let $x_1,\ x_2,\ x_3$ be Cartesian coordinates in reference con£guration of a material body (Lagrangian coordinates), and $X_1,\ X_2,\ X_3$ be Cartesian coordinates of particles of a deformed body (Euler coordinates). Assume that the elastic body has the shape of a cylinder (prism) with arbitrary cross–section in reference con£guration. Cylinder generators are parallel to the x_3–axis, and the coordinates $x_1,\ x_2$ are counted in a cross–section plane. Let's consider the following two–parameteric sets of cylindrical body deformations

$$\begin{aligned} X_1 &= u_1\left(x_1, x_2\right) + l x_3 \qquad\qquad (l, \omega = const) \\ X_2 &= u_2\left(x_1, x_2\right)\cos\omega x_3 - u_3\left(x_1, x_2\right)\sin\omega x_3 \\ X_3 &= u_2\left(x_1, x_2\right)\sin\omega x_3 + u_3\left(x_1, x_2\right)\cos\omega x_3 \end{aligned} \tag{1}$$

$$\begin{aligned} X_1 &= w_1\left(x_1, x_2\right)\cos\psi x_3 - w_2\left(x_1, x_2\right)\sin\psi x_3 \\ X_2 &= w_1\left(x_1, x_2\right)\sin\psi x_3 + w_2\left(x_1, x_2\right)\cos\psi x_3 \\ X_3 &= w_3\left(x_1, x_2\right) + \lambda x_3 \qquad\qquad (\lambda, \psi = const) \end{aligned} \tag{2}$$

The set (1) circumscribes spatial bending of a cylindrical body with rod axis turning into a helical line after the deformation. The axis of this line is orthogonal to the initial axis of a rod. When $l = u_3 = 0$ the formulas (1) describe pure bending of a prismatic beam in the $x_2 x_3$ –plane. The system of equations of nonlinear elastostatics at the absence of mass forces on the deformations (1) is transformed to a system of three scalar equations concerning three functions of two variables $u_k\left(x_1,\ x_2\right)\ (k = 1,\ 2,\ 3)$. The boundary conditions at the load–free lateral surface of a prism for the deformations (1) do not contain the x_3–variable as well. Thus, through substitution (1) the initial spatial problem of nonlinear elasticity is reduced to two–dimensional boundary value problem for ¤at domain in the shape of cross–section of a beam.

The semi–inverse solution (2) represented earlier [2] in other form circumscribes the axial tension–compression and torsion of a prismatic beam.

On the basis of exact relations of the developed torsion theory the direct and inverse Poynting's effects are investigated [5]. These effects describe nonlinear interaction of longitudinal and rotating deformations in prismatic rods. The generalization of the nonlinear theory of torsion of cylindrical bodies laying in the account of isolated and continuously distributed helical dislocations is offered.

The numerical solutions of two–dimensional boundary value problems on a cross–section of a beam are found with the use of variational methods of nonlinear elasticity.

The results listed above are extended to nonlinearly elastic micropolar materials possessing internal rotational degrees of freedom and couple stress. The general three–dimensional nonlinear model of the Cosserat continuum is used [6–8]. In this case semi–inverse solutions yielding reduction of three–dimensional problem to two–dimensional boundary value problems on a cross–section of a beam, describe not only £eld of displacements, but also £eld of microrotations of an elastic medium at £nite deformations.

References

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