

## EXACT TRACKING CONTROL FOR NONLINEAR STRUCTURAL AND MECHANICAL SYSTEMS

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Summary A new methodology for the *exact* tracking control of nonlinear structural and mechanical systems with holonomic and/or nonholonomic trajectory requirements is presented.

We consider in this paper an  $n$  degree-of-freedom structural or mechanical system described by Lagrange's equations [1] as

$$M(x, t)\ddot{x} = F(x, \dot{x}, t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \quad (1)$$

Here,  $x$  is the *generalized* coordinate used in describing the configuration of the system. The  $n$  by  $n$  matrix  $M$  is positive definite and, in general, is a function of  $x$  and the time,  $t$ . We shall assume that the observation  $m$ -vector,  $y(t)$ , is related to the response  $x(t)$  of the system by the observation equation

$$y(t) = Cx(t) \quad (2)$$

where  $C$  is the constant  $m$  by  $n$  measurement matrix. It is desired to control the system described by (1), and determine the control force vector  $F^c(y, \dot{y}, t)$  so that for the controlled system described by,

$$M(x, t)\ddot{x} = F(x, \dot{x}, t) + F^c(y, \dot{y}, t), \quad (3)$$

the measurement  $y(t)$  satisfies the  $s$  desired tracking relations

$$h(y, \dot{y}, t) = 0, \quad (4)$$

where  $h$  is an  $s$ -vector. We assume that the functions  $h_i(y, \dot{y}, t)$  are  $C^1$  and that (4) constitute a set of relations that are feasible for the system to satisfy. The set (4) may contain relations that are integrable and/or non-integrable. Also, the system's initial conditions  $x(0)$  and  $\dot{x}(0)$  are such as to satisfy equation (4). One can relax this assumption on the initial conditions [2]. Differentiating equation (4) with respect to time, one obtains

$$H(y, \dot{y}, t)\ddot{y} = \frac{\partial h}{\partial \dot{y}} \ddot{y}(t) = -\frac{\mathcal{J}h}{\mathcal{J}y} \dot{y} - \frac{\mathcal{J}h}{\mathcal{J}t} \quad (5)$$

where we have denoted the  $s$  by  $m$  matrix  $\frac{\mathcal{J}h}{\mathcal{J}\dot{y}}$  by  $H$ . In view of (2) this can be expressed as

$$B\ddot{x} = HC\ddot{x}(t) = b(y, \dot{y}, t) = b(Cx, C\dot{x}, t), \quad (6)$$

where we have denoted the  $s$  by  $n$  matrix  $HC$  by  $B$ , and the  $s$ -vector  $b$  is given by

$$b(y, \dot{y}, t) = -\frac{\mathcal{J}h}{\mathcal{J}y} \dot{y} - \frac{\mathcal{J}h}{\mathcal{J}t}. \quad (7)$$

We note that by using relations (2) and (4) on the right hand side of (7), the  $s$ -vector  $b$  may be thought of as a known function of  $x$ ,  $\dot{x}$  and  $t$ , as explicitly indicated in (6). From here on, we shall suppress the arguments of the various quantities unless required for clarification. We now state the following central result [2].

**Main Result** The class of control forces that minimize *at each instant of time* the quantity

$$J(t) = (F^c)^T N(x, t) F^c, \quad (8)$$

where the  $n$  by  $n$  matrix  $N(x, t)$  is a given positive definite matrix at each instant of time  $t$ , while causing the controlled system (3) to satisfy relation (6), is given by

$$F^c = N^{-1/2} G^+ (b - BM^{-1}F) \quad (9)$$

where,  $G$  denotes the matrix  $B(N^{1/2}M)^{-1}$ , and  $G^+$  denotes the Moore-Penrose generalized inverse of the matrix  $G$ .

**Proof:** For brevity, the proof will not be given here; it will be presented at the meeting. □

**Example** Consider a nonlinear mechanical system whose equation of motion is given by ( $M = I$ )

$$\ddot{x} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{s}(x_2 - x_1) - c_1 \dot{x} \\ \mathbf{I}x_1 - x_1x_3 - x_2 - c_2 \dot{x}_2 \\ F_3 = x_1x_2 - \mathbf{b}x_3 - c_3 \dot{x}_3 \end{bmatrix} = F, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \quad (10)$$

We shall assume that the measurement  $m$ -vector,  $y$ , is given by the relation

$$y = Cx = \begin{bmatrix} \hat{e}_1 & 1 & 0 \\ \hat{e} & 1 & -1 \\ \hat{e}_1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{u} \\ \dot{u} \end{bmatrix} x, \quad (11)$$

and the desired (and feasible), nonholonomic trajectory we want the system (10) to track is given to be

$$\dot{y}_2 = y_3 \dot{y}_1. \quad (12)$$

We wish to apply a control force,  $F^c$ , so that in the presence of  $F$ , relation (12) is 'exactly' satisfied. For illustration, we take:  $\mathbf{s} = \mathbf{I} = 1$ , and  $\mathbf{b} = 2$ ;  $c_1 = c_3 = 1/2$ , and  $c_2 = 0$ ; and,  $x_1(0) = 0$ ,  $x_2(0) = 1$ ,  $x_3(0) = 1$ ,  $\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$ . Using numerical integration, Figure 1(a) shows the error in the satisfaction of this trajectory requirement, (12), without any control. Using the control force,  $F^c$ , that is explicitly given by equation (9), we see, from Figure 1(b), that this tracking error is reduced to within the numerical precision ( $10^{-8}$ ) with which the equations of motion are integrated.

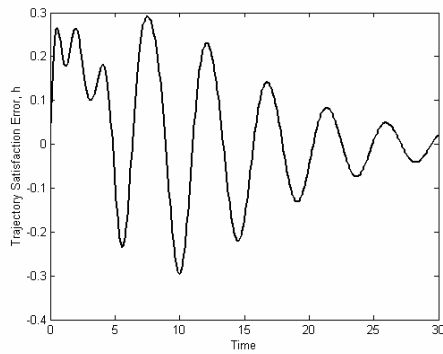


Figure 1(a)

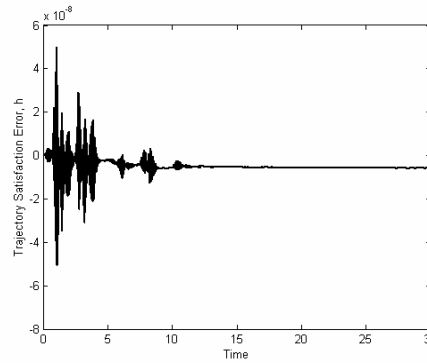


Figure 1(b)

## Conclusions

This paper provides a powerful, new methodology for controlling nonlinear structural and mechanical systems. It has been inspired by recent results in analytical dynamics. The methodology has the following advantages: (1) the control force required to control the *nonlinear* system so that it satisfies the given nonholonomic and/or holonomic trajectory requirements is obtained *explicitly* in closed-form; (2) this control force, theoretically speaking, will cause the trajectory requirements to be 'exactly' satisfied by the nonlinear controlled system; (3) it allows the weighted norm of the control force to be minimized *at each instant of time*--most control methods minimize time integrals of such norms; (4) the computations required to determine the control force involve simple matrix multiplications and additions, making the method attractive for real-time, on-line control of nonlinear mechanical systems. An example of the control methodology is illustrated. It has also been successfully used for precision motion-control of multi-arm robots, for on-orbit formation-flight of satellites, and for controlling flexible, nonlinear vibrating systems.

## References

- [1] Udwadia F. E., Kalaba R. E.: What is the General Form of the Equation of Motion for Constrained Mechanical Systems? *Journal of Applied Mechanics* **69**:335-339, 2002.
- [2] Udwadia F. E.: A New perspective on the Tracking Control of Nonlinear Structural and Mechanical Systems. *Proceedings of the Royal Society of London, Series A* **459**:1783-1800, 2003.