INVERSE OF CONSTITUTIVE EQUATIONS OF ANISOTROPIC HEREDITARY ELASTIC CONTINUA

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Summary *The inverse of Boltzman-Volterra constitutive equations is based on defining the resolvent of the matrix Newman series with the aid of some base properties of the resolvent operators.*

INTRODUCTION

The procedure of inverse of anisotropic hereditary elastic continua constitutive equations from the mathematical point of view is a solution of the Volterra integral equation system of the second kind. The results of the inverse may be used when describing anisotropy of rheological properties or when using the Volterra correspondence principle [1] to solve the boundary problems.

INITIAL CONSTITUTIVE EQUATIONS

The constitutive equations may be represented in the following matrix form [2]

$$\{\overrightarrow{\varepsilon}\} = [A + \Lambda^*]\{\overrightarrow{\sigma}\}, \tag{1}$$

where $[A]$ and $[\Lambda^*]$ are the matrices of elastic compliances and creep operators. The procedure of the inverse is carried out assuming that resolvent operator has the same type and parameters, i.e. $[\Lambda^*] = [\Lambda]R^*(\mu)$, where $R^*(\mu)$ is a resolvent operator. Correctness of the approach is discussed in [1, 3]. Constitutive equations of anisotropic hereditary media (1) may be rewritten as follow [2, 4]

$$\varepsilon_i = a_{ij}\sigma_j + \lambda_{ij}\int_0^t R(\mu, t-\tau)\sigma_j(\tau)\tau d\tau, \tag{2}$$

where $\int_0^t R(\mu, t-\tau)\sigma_j(\tau)d\tau = R^*(\mu)\cdot\sigma_j$.

PROCEDURE OF INVERSE

Taking into account the previous assumptions the inverse of the operator matrix can be represented as

$$[A + \Lambda^*]^{-1} = [I + A^{-1}\Lambda R^*(\mu)]^{-1}[A]^{-1}, \tag{3}$$

where I is a unit matrix.
Denoting $A^{-1}\Lambda = N$ and taking expansion the right-hand side (3) into series $[I + NR^*(\mu)]^{-1} = I - NR^*(\mu) + (NR^*(\mu))^2 - (NR^*(\mu))^3 + \ldots$, Newman series to the matrix resolvent $[\tilde{R^*}]$ yields

$$[\tilde{R^*}] = R^*(\mu)[I - NR^*(\mu) + (NR^*(\mu))^2 - \ldots] = R^*(\mu)\sum_{l=0}^{\infty}(-NR^*(\mu))^l. \tag{4}$$

Performing diagonalization of matrix $N = Xdiag(\lambda_1, \ldots \lambda_n)X^{-1}$ and using that $N^l = Xdiag(\lambda_1^l, \ldots \lambda_n^l)X^{-1}$ sum in (4) can be represented as

$$\sum_{l=0}^{\infty}(-NR^*(\mu))^l = Xdiag(\sum_{l=0}^{\infty}(-\lambda_1 R^*(\mu))^l, \ldots, \sum_{l=0}^{\infty}(-\lambda_n R^*(\mu))^l)X^{-1}. \tag{5}$$

Substituting (5) in the right-hand side (4), the resolvent of Newman series matrix becomes

$$[\tilde{R^*}] = Xdiag(R^*(\mu)\sum_{l=0}^{\infty}(-\lambda_1 R^*(\mu))^l, \ldots, R^*(\mu)\sum_{l=0}^{\infty}(-\lambda_n R^*(\mu))^l)X^{-1} \tag{6}$$

Diagonal elements of (6) are the resolvents of the operator $R^*(\mu)$ respectively and taking into account known property of resolvent operators [1] $R^*(\mu)\sum_{l=0}^{\infty}(-\lambda_k R^*(\mu))^l = \tilde{R^*_{\lambda_k}}$ yields

$$[\tilde{R^*}] = Xdiag(\tilde{R^*_{\lambda_1}}, \ldots, \tilde{R^*_{\lambda_n}})X^{-1} \tag{7}$$

Thus to derive resolvent of the Newman series matrix we need to derive the resolvent operators corresponding to eigenvalues of matrix $[N]$. The inverse system of constitutive equations is

$$[A + \Lambda R^*(\mu)]^{-1} = [I - \tilde{R^*}]C, \tag{8}$$

where $[C] = [A]^{-1}$ stiffness matrix. Using the relationship [1, 2] that resolvent of resolvent operator is the same operator having the shift of the parameter $\tilde{R^*_\lambda}(\mu) = R^*(\mu - \lambda)$, the inverse of the constitutive equations can be obtained from (8)

$$\{\overrightarrow{\sigma}\} = [C - P^*]\{\overrightarrow{\varepsilon}\}, \tag{9}$$

where $[P^*] = X diag(\lambda_1 R^*(\mu - \lambda_1), \ldots, \lambda_n R^*(\mu - \lambda_n) X^{-1} C$.
It is obvious that at $t = 0$ the system (9) converts to equations of anisotropic elasticity theory. Elements of $[C - P^*]$ matrix are long term moduli of the hereditary media.

APPLICATION OF THE PROCEDURE

As a demonstration of the inverse procedure the constitutive equations describing plane stress state of glass fiber reinforced plastic KAST [1] were taken

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{E_1}\left[1 + k_1 \epsilon^*_\alpha(-\beta)\right] \cdot \sigma_1 - \frac{\nu_1}{E_2}\sigma_2 \\
\varepsilon_2 &= -\frac{\nu_2}{E_1}\sigma_1 + \frac{1}{E_2}\left[1 + k_2 \epsilon^*_\alpha(-\beta)\right] \cdot \sigma_2 \\
\gamma_{12} &= \frac{1}{G}\left[1 + k_6 \epsilon^*_\alpha(-\beta)\right] \cdot \tau_{12},
\end{aligned} \tag{10}$$

where $\epsilon^*_\alpha(-\beta)$ Rabotnov's fraction-exponential operator [1], combining Abel's operator singularity and exponential boundedness, $\epsilon^*_\alpha(-\beta) \cdot \sigma = \int_0^t \epsilon_\alpha(-\beta, t-\tau) \cdot \sigma(\tau) d\tau$, where $\epsilon_\alpha(-\beta, t) = t^\alpha \cdot \sum\limits_{n=0}^{\infty} \frac{(-1)^n \left(\beta \cdot t^{1+\alpha}\right)^n}{\Gamma[(1+\alpha)(1+n)]}$, $-1 < \alpha < 0$ – kernel singularity parameter. Matrices $[A]$ and $[\Lambda]$ corresponding to constitutive equations (2) are

$$\mathbf{A} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_1}{E_2} & 0 \\ -\frac{\nu_2}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G} \end{pmatrix} \quad \mathbf{\Lambda} = \begin{pmatrix} \frac{k_1}{E_1} & 0 & 0 \\ 0 & \frac{k_2}{E_2} & 0 \\ 0 & 0 & \frac{k_6}{G} \end{pmatrix}$$

The numerical values of constitutive equations (10) are [1]: $E_1 = 17900$ MPa, $E_2 = 18680$ MPa, $G = 3210$ MPa, $\nu_1 = 0.195, \nu_2 = 0.187$, $k_1 = 0.05, k_2 = 0.07$, $k_6 = 0.137$, $\alpha = -0.8$, $\beta = 0.1$. Parameters k_1, k_2, k_6 and β were defined that the unit of time was an hour. Following the scheme of the calculations we readily can get the inverse constitutive equations similar to (9). Using asymptotical property of Rabotnov's fraction exponential operator [1] $\epsilon^*_\alpha(-\beta) \cdot 1_{|t=\infty} = 1/\beta$ we can verify the procedure of the calculations. An obvious way of the verification is to compare asymptotical values of the initial $(A + \Lambda/\beta)^{-1}$ and inverse $(C - P^*_\infty)$, where $P^*_\infty = X diag(\frac{\lambda_1}{\lambda_1+\beta}, \frac{\lambda_2}{\lambda_2+\beta}, \frac{\lambda_3}{\lambda_3+\beta}) X^{-1} C$, matrices. The coincidence of the matrices values with round-off errors was shown.

CONCLUSIONS

An analytical method of inverse of hereditary anisotropic elastic continua constitutive equations of has been presented. The procedure of the inverse is reduced to matrix calculations and does not depend on the kind of the resolvent operator. A kernel of the operator may be chosen either exponential, power (Abel's), fraction-exponential (Rabotnov's) functions. The procedure may be extended to inverse of constitutive equations with different resolvent operators.

References

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