

DYNAMICS OF BECK'S COLUMN VIA NONLINEAR NORMAL MODES

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Summary The present work deals with the nonlinear dynamic response of Beck's column under a partial follower loading at its tip, characterised by a non-conservativeness parameter η . In the context of exact Elastica, the beam's motion is governed by a strongly nonlinear integro-differential equation with respect to the cross-sectional rotation. In order to capture all nonlinear phenomena associated with the motion in both divergence and flutter regimes, this equation is discretized using a two-mode generalized Galerkin approach, in which the spatial functions are nonlinear normal modes satisfying all the boundary conditions. In doing this and after numerous symbolic manipulations the equation of motion is reduced to a set of two second order differential equations with respect to the time functions. These are solved numerically and the corresponding behaviour in both regions of existence and non-existence of adjacent equilibria, i.e. divergent motion or limit cycles (flutter) are fully assessed.

INTRODUCTION - PROBLEM DESCRIPTION

Various analytical and numerical methods have been developed for analyzing the dynamics of continuous systems of finite special extent. There are two general classes of methodologies for computing these dynamics; the first one is based on the discretization of the governing partial differential equation while the second one relies on the direct analysis of the nonlinear equation of motion, without involving any discretization schemes[1]. Methods based on discretization express the displacement $u(x,t)$ in the following series form:

$$u(x,t) = \sum_{i=1}^n \phi_i(x_i) q_i(t) \quad (1)$$

where one of the functions $\phi_i(x_i)$ or $q_i(t)$ is prescribed. A basic limitation of this type of analysis is that, in cases where $\phi_i(x_i)$ are prescribed as the normal modes of the associated linear system, the mode shape of the nonlinear vibration is defined *a priori*, and thus, is unaffected by the overall nonlinearity. However, this method may be proven quite sufficient and produces reliable results, if the functions $\phi_i(x_i)$ chosen are nonlinear normal modes, that contain by nature the nonlinearities of the problem and in case of nonconservative systems the cause of nonconservativeness itself. One of the most simple and simultaneously popular nonconservative flexible structural elements, dealt with extensively in the literature is the cantilever beam under partial follower loading at its tip, known as the column of Beck [2]. The objective of the present work is the computation of the nonlinear dynamics of this particular system, using a two-mode discretization procedure with a proper choice of the shape functions, as described above, within the context of exact Elastica (large amplitude vibration) analysis.

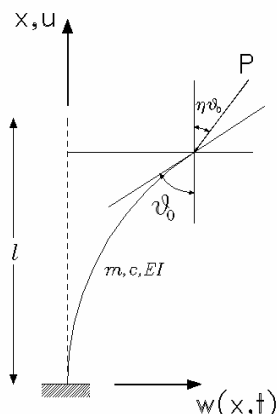


Fig.1. Geometry and sign convention of Beck's column

In doing this, we consider the uniform cantilever beam shown on Figure 1, made of linearly elastic material, subjected to a partial follower loading P at its tip. This is characterized by the nonconservativeness parameter η , while I , m and c are the moment of inertia of the beam's cross-section, the linear density and the viscous damping coefficient per unit length respectively. Introducing the dimensionless quantities

$$\bar{x} = \frac{x}{l}, \quad k^2 = \frac{Pl^2}{EI}, \quad \phi(\bar{x}, \tau) = \theta(x, t), \quad \tau = \sqrt{\frac{EI}{ml^4}} t, \quad c^* = \frac{l^2}{\sqrt{mEI}} c, \quad \bar{w}(\bar{x}, \tau) = \frac{w(x, t)}{l}$$

and assuming incompressibility of the column's material, the motion of the beam is governed by the following strongly nonlinear non-dimensionalized integro-differential equation [3]:

$$\begin{aligned} \phi'''' + \int_0^{\bar{x}} [\ddot{\phi} \cos \phi - \dot{\phi}^2 \sin \phi] d\xi + k^2 \phi' \cos \phi \cos \eta \phi_0 - \\ \frac{1}{2} \phi' \cos \phi \int_0^{\bar{x}} \int_{\bar{x}}^l [\ddot{\phi} \sin 2\phi + 2\dot{\phi}^2 \cos 2\phi] d\xi d\xi - \\ - \frac{1}{2} \sin \phi \int_{\bar{x}}^l [\ddot{\phi} \sin 2\phi + 2\dot{\phi}^2 \cos 2\phi] d\xi + c^* \int_0^{\bar{x}} \dot{\phi} \cos \phi d\xi = 0 \end{aligned} \quad (2)$$

associated with the boundary conditions given below:

$$\phi(0, \tau) = 0, \quad \bar{w}(0, \tau) = \int_0^{\bar{x}} \sin \phi(\xi, \tau) d\xi \Big|_{\bar{x}=0} = 0, \quad \phi'(l, \tau) - k^2 \phi_0 \cos \eta \phi_0 + k^2 \sin \eta \phi_0 = 0 \quad (3)$$

where $\phi_0 = \phi(l, \tau)$, while zero initial conditions are considered.

Thereafter, both equation (2) and conditions (3) are expressed in terms of $\bar{w}(\bar{x}, \tau)$ by employing the approximate relation

$$\phi \approx w' \left(1 + \frac{w'^2}{6} \right) \quad (4)$$

while the trigonometric functions involved are substituted by their Padé approximations of $O[h^8]$, by means of modern commercial mathematical software. Thus, the integro-differential equation of motion is transformed into a partial differential equation with respect to $\bar{w}(\bar{x}, \tau)$, while similar transformations are performed on the boundary conditions.

The resulting relations are lengthy and for reasons of brevity are not given herein.

TWO-MODE DISCRETIZATION USING NONLINEAR NORMAL MODES

The partial differential equation of motion is discretized according to (1) using an assumed two-mode solution of the form $\bar{w}(\bar{x}, \tau) = s_1(\tau)g_1(\bar{x}) + s_2(\tau)g_2(\bar{x})$, where functions $g_i(\bar{x})$ are nonlinear normal modes satisfying all boundary conditions and thus directly including the problem's nonlinear as well as nonconservative nature. These functions, depending explicitly on the nonconservativeness parameter η , are taken equal to

$$g_i(x) = 28 - \cos\left[\frac{3\pi x}{2}\right] - 27 \cos\left[\frac{\pi x}{2}\right] + \alpha_i x^2 (x-3) \quad (i=1,2) \quad (5)$$

where coefficients α_i are computed symbolically, being extremely complicated nonlinear functions of η . Their shapes however were found in excellent agreement with those of the relevant literature [4], in which nonlinear normal modes were computed through an invariant manifold approach. After a series of symbolic integrations, the spatial coordinate in the differential equation of motion is eliminated and a system of two second order strongly nonlinear differential equations with respect to functions $s_1(\tau)$ and $s_2(\tau)$ is reached. These are solved numerically for every desired combination of the foregoing parameters and the dynamic behavior of the system dealt with is assessed.

NUMERICAL RESULTS AND CONCLUSIONS

Following the procedure outlined above, one can establish the dynamic response of the nonconservative system dealt with in both regions of existence ($\eta < 0.50$) or non-existence ($\eta > 0.50$) of adjacent equilibria, i.e. for both divergence and flutter areas. Typically, within the region of divergence instability and below a certain critical value of the loading the system's motion is stable and associated with a point attractor response at the origin, as depicted in Figure 2a.

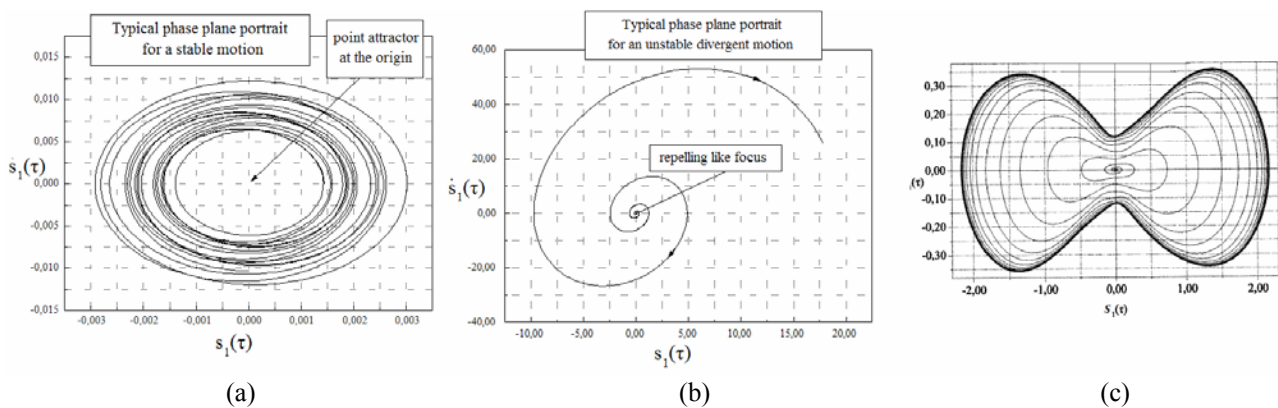


Fig.2. Typical phase planes of three different types of dynamic behavior

Beyond this value the origin acts as a repelling like focus, leading to an unbounded divergent motion, shown in Figure 2b, while for much higher values of the loading the motion is associated with stable limit cycles (flutter), perceived in Figure 2c. On the other hand, for $\eta > 0.50$ the resulting motions are either stable, as in Figure 1a, or typically of a flutter type, associated with large amplitude isolated periodic orbits. From a qualitative viewpoint, the results of the present study match the ones obtained using totally different approaches [2], a fact fully validating the proposed methodology, which may be extended to the study of other continuous nonconservative nonlinear systems, provided that it will be properly adjusted to fit their specific requirements.

References

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