NASH EQUILIBRIUM IN BICRITERIA STRUCTURAL OPTIMIZATION

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Summary Nash equilibrium, a game theoretic concept, has lately been introduced in multicriteria optimization of structures as a computationally cheap alternative to Pareto optima. In this paper we formulate a bicriteria optimization problem game theoretically and present a class of problems for which there exists a Pareto optimal Nash equilibrium. We illustrate the theory with a static truss optimization example.

INTRODUCTION

Pareto optimality is the key concept of multicriteria optimization. The set of Pareto optima is, however, usually expensive to compute. For this reason some researchers [1] have lately introduced the Nash equilibrium as a computationally cheap alternative to the Pareto optima. The Nash equilibrium is a non-cooperative game theoretic solution concept whereas the multicriteria optimization problem is like a cooperative game. Thus, a Nash equilibrium point as a solution to a multicriteria optimization problem can be dominated, that is, it is not necessarily Pareto optimal. In addition, a Nash equilibrium point, in contrast to a Pareto optimal point, need not exist even in well-posed problems. In this paper we formulate a bicriteria structural optimization problem game theoretically and present a class of problems where a Pareto optimal Nash equilibrium exists.

BICRITERIA OPTIMIZATION PROBLEM AND PARETO OPTIMALITY

We define the bicriteria optimization problem as follows

$$\min_{\mathbf{x}\in\Omega} \mathbf{f}(\mathbf{x}) = \left[f_1(\mathbf{x}),\ f_2(\mathbf{x})\right]^{\mathrm{T}} \tag{1}$$

where $\mathbf{f} : \Omega \to \mathbf{R}^2$ is a vector objective function whose components are functions from Ω to $\mathbf{R}$ to be minimized simultaneously (criteria). Here Ω is a feasible set which we assume to be of form $\Omega = \{\mathbf{x}\in\mathbf{R}^n \mid \mathbf{g}(\mathbf{x}) \le \mathbf{0}\}$, where $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^k$ is constraint vector function. Finally, the image of Ω under $\mathbf{f}$ is called an attainable set denoted as $\Lambda = \mathbf{f}(\Omega)$. Next, we define the key solution concept, the Pareto optimality for problem (1).

Definition 1 A vector $\mathbf{x}^*\in\Omega$ is Pareto optimal if and only if there exist no other vector $\mathbf{x}\in\Omega$ such that $f_i(\mathbf{x}) \le f_i(\mathbf{x}^*)$ for $i = 1,2$ with strict inequality for at least one i. We call $\mathbf{f}(\mathbf{x}^*)\in\Lambda$ a minimal point.

Problem (1) has at least one Pareto optimum if the criteria are smooth and Ω is compact [2].

GAME THEORETIC FORMULATION AND NASH EQUILIBRIUM

When formulating (1) as a non-cooperative two-player game, we assign each player i one objective function f_i, a payoff. Player i wants to minimize his payoff f_i as a function of his strategy $\mathbf{x}_i \in \mathbf{R}^{n_i}$. We denote by $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)\in\mathbf{R}^n$ a strategy profile ($n = n_1 + n_2$). Then the set of feasible strategy profiles is the feasible set of optimization problem (1), $\Omega \subset \mathbf{R}^n$. Now the game formulation of (1) is as follows.

$$\text{player } i\ (i = 1,2):\ \min_{\mathbf{x}_i} f_i(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x}\in\Omega \tag{2}$$

This formulation is not unique but can be done in different ways depending on the dimensions of Ω. In case $\mathbf{x} = (x_1, x_2)\in\Omega\subset\mathbf{R}^2$ these formulations are

$$\text{I}:\ \text{player 1}: \min_{x_1} f_1(x_1,x_2)\ \&\ \text{player 2}: \min_{x_2} f_2(x_1,x_2),\quad \text{II}:\ \text{player 1}: \min_{x_2} f_1(x_1,x_2)\ \&\ \text{player 2}: \min_{x_1} f_2(x_1,x_2) \tag{3}$$

These formulations represent different games in the sense they may have different non-cooperative game solutions, e.g. Nash equilibrium points which we define next.

Definition 2 A strategy profile $\mathbf{x}^* = (\mathbf{x}_1^*, \mathbf{x}_2^*)\in\Omega$ is a Nash equilibrium for game (2) if and only if for all $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)\in\Omega$ it holds that $f_1(\mathbf{x}_1^*, \mathbf{x}_2^*) \le f_1(\mathbf{x}_1, \mathbf{x}_2^*)$ and $f_2(\mathbf{x}_1^*, \mathbf{x}_2^*) \le f_2(\mathbf{x}_1^*, \mathbf{x}_2)$.

As well known, the existence of Nash equilibrium is more complicated issue than that of Pareto optimality. Moreover, the Nash equilibrium tends to be Pareto inefficient [3]. However, there is a class of problems for which there exists a Pareto optimal Nash equilibrium.

Theorem 1 Let $\Omega = [a,b]\times[c,d] \in \mathbf{R}^2$. Let $f_1, f_2 \in C^0(\Omega)$ be such that functions $x_i \mapsto f_j(x_1, x_2)$ $(i, j = 1,2)$ are strictly monotonous for all $\mathbf{x} = (x_1, x_2) \in \Omega$. Finally, let $\mathbf{f} = [f_1, f_2]^{\mathrm{T}}$ be bijection. Then either for game I or game II there exists a Pareto optimal Nash equilibrium point.

Proof. Straightforward and needs too much space to be included.

A STATIC TRUSS EXAMPLE

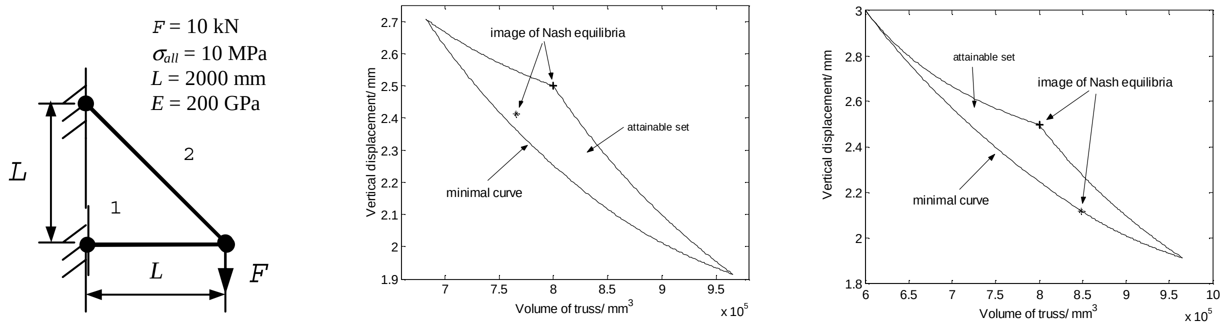
Consider a two-bar isostatic truss under a static vertical loading illustrated below (Fig. 1). Our bicriteria optimization problem is

$$\begin{aligned} &\min[V(\mathbf{x}), \Delta(\mathbf{x})]^{\mathrm{T}} \\ &\text{s.t.:} \quad F/\sigma_{sall} \le x_1 \le \bar{x}_1\,, \quad \sqrt{2}F/\sigma_{sall} \le x_2 \le \bar{x}_2 \end{aligned} \tag{4}$$

where $\mathbf{x} = (x_1, x_2)$ is the design vector consisting of areas of the truss members, V is the volume of the truss, and Δ is the vertical displacement of the joint of the truss members. The objective functions are $V(\mathbf{x}) = L\left(x_1 + \sqrt{2}x_2\right)$ and $\Delta(\mathbf{x}) = \frac{FL}{E}\left(\frac{1}{x_1} + \frac{2\sqrt{2}}{x_2}\right)$. These functions are strictly monotonous in the sense of theorem 1 when variables are non-zero. First, let the upper limit for both of the bars of the truss be 200 mm^2. Then the feasible set for this problem is $\Omega_1 = [100, 200]\times[\sqrt{2}\cdot 100, 200]\,\mathrm{mm}^2$. The Nash equilibria in games I and II are $\mathbf{x}_{\mathrm{I}} = (100, 200)$ and $\mathbf{x}_{\mathrm{II}} = (200, \sqrt{2}\cdot 100)$, respectively, and both of them are Pareto inefficient (see Fig. 1). The reason for the Pareto inefficiency of $\mathbf{x}_{\mathrm{I}}$ is that $\mathbf{f}$ is not bijection on Ω_1. The determinant of $\nabla\mathbf{f}$ vanishes on the line $x_2 = \sqrt{2}x_1$ and there are points on this line that dominates $\mathbf{x}_{\mathrm{I}}$. Thus we have to assume bijectivity for theorem 1 to hold generally.

Second, let the feasible set be $\Omega_2 = [\sqrt{2}\cdot 100, 200]\times[\sqrt{2}\cdot 100, 200]$. Now the Nash equilibria of games I and II are $\mathbf{x}_{\mathrm{I}} = (\sqrt{2}\cdot 100, 200)$ and $\mathbf{x}_{\mathrm{II}} = (200, \sqrt{2}\cdot 100)$, respectively. But this time $\mathbf{x}_{\mathrm{I}}$ is Pareto optimal even though it is the only point in Ω_2 where $\nabla\mathbf{f}$ has no inverse.

We see from figure 1 that in both cases the point $\mathbf{x}_{\mathrm{II}}$ is the worst point on Ω in the sense of Pareto optimality.



(a) (b) (c)

Figure 1 Idealized structure and design data of static truss example (a), attainable sets, image of Nash equilibria and minimal curves of the two-bar static truss example for Ω_1 (b) and Ω_2 (c), characters * and + indicate $\mathbf{f}(\mathbf{x}_{\mathrm{I}})$ and $\mathbf{f}(\mathbf{x}_{\mathrm{II}})$

CONCLUSIONS

The game formulation of bicriteria optimization problem can be done in number of ways depending on the dimension of the feasible set. These formulations may have different Nash equilibrium points and some of these points can be the worst in sense of Pareto efficiency. However, if the conditions of theorem 1 are met, the Nash game approach gives Pareto optimal Nash equilibrium point through solving the two game formulations by using, e.g., the fixed point iteration.

References

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