A TECHNIQUE FOR NONSMOOTH OPTIMIZATION BASED ON THE INTERIOR POINT FEASIBLE DIRECTIONS ALGORITHM

José Herskovits*, Wilhelm Passarella**, Susana Scheimberg***, Regina Burachik***
* *COPPE/Federal University of Rio de Janeiro, Mechanical Engineering Program, Caixa Postal 68503, 21945 970 Rio de Janeiro, Brazil*
** *Federal University of Juiz de Fora, Department of Mathematics, Juiz de Fora, Brazil*
*** **COPPE/Federal University of Rio de Janeiro, Systems Engineering Program, Rio de Janeiro, Brazil*

Summary A new method for the minimization of nonsmooth convex functions is presented. This approach defines a constrained optimization Equivalent Problem and a sequence of Approximate Problems, where the constraints are substituted by tangent planes. At each iteration a Feasible Direction for the Equivalent Problem is obtained by computing Feasible Descent Directions for the Approximate Problems. The present approach is based on FD_IPA, a Feasible Directions Interior Point Algorithm for constrained smooth optimization. We prove global convergence and solve very efficiently several test problems.

INTRODUCTION

Nonsmooth functions are frequently present in Structural Optimization. This is the case with applications involving eigenvalues of matrices depending of the design variables. Smooth optimization algorithms cannot be applied to this problems because they usually stop at non optimal solutions.

We consider the the optimization problem:

$$(OP) \quad \min_{x} F(x)$$

where $x \in R^n$ and F is convex and not necessarily differentiable. We call $\partial F(x)$ the Subdifferential of F at x. Any vector $s \in \partial F(x)$, $s \in R^n$, is a Subgradient, [2]. In general, nonsmooth optimization methods assume that there is an available tool that, for a given x, computes $F(x)$ and one subgradient $s(x) \in \partial F(x)$.

Let us define the auxiliary variable $z \in R$ and the constrained Equivalent Problem:

$$(EP) \begin{cases} \min\limits_{x,z} z \\ \text{s. t. } F(x) \le z \end{cases} \tag{1}$$

The present algorithm builds a sequence (x^k, z^k) at the epigraph of $F(x)$ such that $z^{k+1} < z^k$. At each iterate (x^k, z^k), a sequence of Approximate Problems is constructed, with the constraints of (EP) substituted by tangent planes. A Search Direction for the equivalent problem is obtained by computing a Feasible Descent Direction for the current approximate problem. If the step length is "short", the approximate problem is updated by adding a new tangent plane and a new search direction is computed. This procedure is repeated until a "good" step is obtained. When this happens, the search direction is a Feasible Descent Direction of the equivalent problem. We employ the Feasible Directions Interior Point Algorithm for constrained smooth optimization to compute the search directions, [1].

We prove global convergence and describe the numerical results with a set of test problems. The present method behaves very well when compared to other techniques and is very strong, being all test problems solved with the same set of parameters.

ABOUT FD_IPA, THE FEASIBLE DIRECTIONS INTERIOR POINT ALGORITHM

Some aspects of the numerical implementation of the Feasible Directions Interior Point Algorithm which are of interest in the present paper, are described now. A complete description of FD_IPA and the corresponding theoretical and numerical results can be found in [1].

Let be the smooth inequality constrained optimization problem:

$$\begin{cases} \min\limits_{x} f(x) \\ \text{s. t. } g(x) \le 0 \end{cases} \tag{2}$$

where $x \in R^n$ and $g \in R^m$. To get a Feasible Descent Direction of the problem at a point x, FD_IPA solves two linear systems with the same matrix:

$$\begin{bmatrix} B & \nabla g(x) \\ \Lambda \nabla g^T(x) & G(x) \end{bmatrix} \begin{bmatrix} d_0 & d_1 \\ \lambda_0 & \lambda_1 \end{bmatrix} = - \begin{bmatrix} \nabla f(x) & 0 \\ 0 & \lambda \end{bmatrix} \tag{3}$$

where $G(x) \equiv diag[g(x)]$ and $\Lambda \equiv diag(\lambda)$ and $B \in R^{n\times n}$ is a symmetric and positive definite matrix. In particular B can be taken equal to the second derivative of the Lagrangian function or to a quasi-Newton approximation.

In [1], it was proved that the vector $d_0 \in R^n$ is a descent direction of the objective function, that is, $d_0^t \nabla f(x) < 0$. However, d_0 is not always a feasible direction of the problem. We also have that $d_1 \in R^n$, called restoring direction, points to the interior of the feasible set. FD_IPA obtaines a Feasible Descent Direction by computing $d = d_0 + \rho d_1$ for $\rho > 0$ and $\rho > (\alpha - 1) d_0^t \nabla f(x) / d_0^t \nabla f(x)$, $\alpha \in (0, 1)$. Since $d_0^t \nabla f(x) < 0$ it is easy to prove that $d^t \nabla f(x) < 0$. Finally, FD_IPA carries out a line search on d looking for a new interior point with a lower objective.

AN ALGORITHM FOR NONSMOOTH OPTIMIZATION

The iterative process for present algorithm for nonsmooth optimization is presented now.

Data: Define an initial iterate (x^0, z^0) such that $z^0 > F(x^0)$ and set $k = 0$.
Step 1. Define, for the iteration k, the initial sub-iterate $x_0^k \equiv x^k$ and set $l = 0$.
Step 2. Define the Approximated Problem:

$$(AP_l^k) \begin{cases} \min\limits_{x,z} z \\ \text{s. t. } F(x_0^k) + s(x_0^k)(x - x_0^k) \le z \\ \quad\quad F(x_1^k) + s(x_1^k)(x - x_1^k) \le z \\ \quad\quad \ldots\ldots\ldots\ldots\ldots \\ \quad\quad F(x_l^k) + s(x_l^k)(x - x_l^k) \le z \end{cases} \tag{4}$$

where $s(x_l^k) \in \partial F(x_l^k)$. The constraints are linear and tangent to $z = F(x)$ at $x_0^k, x_1^k, ..., x_l^k$ respectively.
Step 3. Search Direction:
Compute $d_l^k \equiv (d_{lx}^k, d_{lz}^k)$, a Feasible Descent Direction given by FD_IPA at (x^k, z^k) when solving (AP_l^k).
Step 4. Line Search:
Find $t_l^k = sup(t;$ such that $g(x + td_x, z + td_z) \le 0)$
If $t_l^k > 1$ set $t_l^k = 1$
Step 5. Stopping test for sub-iterations
If

$$F(x^k + \mu t_l^k d_{lx}^k) \le z^k + \mu t_l^k d_{lz}^k$$

the search direction is good. Then, set

$$(x^{k+1}, z^{k+1}) = (x^k + \mu t_l^k d_{lx}^k, z^k + \mu t_l^k d_{lz}^k)$$

Increase iterations counter: $k = k + 1$ and Return to **Step 1.**
Step 6. Define a new sub-iterate:

$$x_{l+1}^k = x^k + \mu t_l^k d_{lx}^k$$

Increase sub-iterations counter: $l = l + 1$ and Return to **Step 2.** □

The present method consists on a two-levels iterative process. For each iterate (x^k, z^k) the sequence of sub-iterates $\{(x_l^k, z_l^k)\}$ is finite. The constraints of (AP_l^k) are taken tangent to $F(x)$ at $\{x_l^k\}$. The sub-iterations stop when the new point $(x^k + \mu t_l^k d_{lx}^k, z^k + \mu t_l^k d_{lz}^k)$ is feasible.

CONCLUSIONS

The present algorithm is simple to code and includes a few number of parameters. The numerical results show that it is very strong, since all the test problems were solved efficiently with the same set of parameters. Complete theoretical results support the present technique.

References

[1] Herskovits J.: Feasible Directions Interior Point Technique For Nonlinear Optimization. *JOTA, Journal for Optimization Theory and Applicationa* **v99-1**, pp. 121-146, 1998.
[2] Hirriart-Urruty, J. B. and Lemarechal, C.: *Convex Analysis and Minimization Algorithms*. Springer Verlag, Heidelberg, 1993.