Mohr-Coulomb Yield Criterion for Cosserat Continua

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Summary The classical yield criteria cannot be applied directly to Cosserat continua since strain tensor and stress tensor in Cosserat continua are asymmetric. In this contribution strain tensor and stress tensor are decomposed into symmetric and skew-symmetric components, and the formulae of stress invariants are modified for Cosserat continua. A Mohr-Coulomb yield criterion for Cosserat continua is established, and a numerical example demonstrates the validity of this formulation.

Instroduction

The classical continuum suffers from pathological mesh-dependence and losing ellipticity of governing field equations when strain-softening models are employed in failure analyses. The Cosserat continuum as a model, which incorporates an internal length scale, can be used to solve such a problem in substance. Recently, scholars such as Muhlhaus and Borst, have made significant contributions to the development of the Cosserat theory. Because they paid more attention to the superior performance of the Cosserat continuum compared with the classical continuum, a criterion, which was analogous to von Mises yield criterion for the classical continuum, was applied in their research for the simplicity. However, many materials in geotechnial engineering cannot be modeled by von Mises yield criterion. Therefore, it is necessary to establish Mohr-Coulomb and other traditional yield criteria for Cosserat continua.

MOHR-COULOMB YIELD CRITERION FOR COSSERAT CONTINUA

In the Cosserat continuum, a geometrically independent rotation vector ω is introduced besides the displacement vector $\boldsymbol{u}$ in the classical continuum for plane problems. One curvature tensor $\boldsymbol{\kappa}$ and one couple tensor $\boldsymbol{m}$ are brought forward likewise.

Introduce the stain vector $\boldsymbol{\varepsilon}$ and the stress vector $\boldsymbol{\sigma}$, as shown in Fig.1. Let l be the characteristic length, and it is an important parameter in the Cosserat continuum. If l tends to zero, the Cosserat continuum will reduce to the classical continuum.

$$\boldsymbol{\varepsilon}=\{\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad \varepsilon_{12} \quad \varepsilon_{21} \quad \kappa_{13}l \quad \kappa_{23}l\}^{T}, \tag{1}$$

$$\boldsymbol{\sigma}=\{\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{21} \quad m_{13}/l \quad m_{23}/l\}^{T}. \tag{2}$$

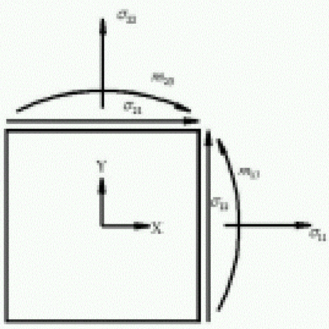

Fig. 1 A unit of Cosserat continua

In the classical continuum, the yield functions are commonly expressed by the three stress invariants J_1, J_2, J_3, and the internal variable γ as follows:

$$f(\sigma_{ij},\gamma)=f(J_1,J_2,J_3,\gamma)\le 0. \tag{3}$$

Because the strain vector and the stress vector are asymmetric, some modification should be made in order to adapt them for Cosserat continua. The strain tensor and the stress tensor are decomposed into symmetric and skew-symmetric components as Findeiss does,

$$\varepsilon_{\alpha\beta}^{sym}=\frac{1}{2}(u_{\beta,\alpha}+u_{\alpha,\beta}), \quad (4) \qquad \varepsilon_{\alpha\beta}^{ant}=\frac{1}{2}(u_{\beta,\alpha}-u_{\alpha,\beta})+e_{\beta\alpha}\omega, \quad (5) \qquad \kappa_{\alpha 3}=\omega_{,\alpha}, \quad (6)$$

$$\sigma_{\alpha\beta}^{sym}=D_{\alpha\beta\gamma\delta}^{sym}\varepsilon_{\gamma\delta}^{sym}, \quad (7) \qquad \sigma_{\alpha\beta}^{ant}=G_c\varepsilon_{\alpha\beta}^{ant}, \quad (8) \qquad m_{\alpha 3}=G\kappa_{\alpha 3}, \quad (9)$$

where $D_{\alpha\beta\gamma\delta}^{sym}$ is the moduli tensor for the symmetric components, and G_c is the Cosserat shear modulus. The three stress invariants can be modified by introducing the stress symmetric and skew-symmetric components,

$$J_1=\sigma_{11}+\sigma_{22}+\sigma_{33}, \quad (10) \qquad J_2=J_2^{sym}+J_2^{ant}+J_2^{m}, \quad (11) \qquad J_3=\frac{1}{3}(s_{11}^{sym}s_{22}^{sym}s_{33}^{sym}-s_{12}^{sym}s_{21}^{sym}s_{33}^{sym}), \quad (12)$$

$$J_2^{sym}=\frac{1}{2}[(s_{11}^{sym})^2+(s_{22}^{sym})^2+(s_{33}^{sym})^2+(s_{12}^{sym})^2+(s_{21}^{sym})^2], \tag{13}$$

$$J_2^{ant} = \frac{1}{2}[(s_{12}^{ant})^2 + (s_{21}^{ant})^2], \quad (14) \qquad J_2^{m} = \frac{1}{l^2} m_{13} m_{23}, \quad (15) \qquad s_{\alpha\beta}^{sym} = \sigma_{\alpha\beta}^{sym} - \frac{1}{3} J_1 \delta_{\alpha\beta}, \ \ s_{\alpha\beta}^{ant} = \sigma_{\alpha\beta}^{ant}. \quad (16)$$

Suppose that the form of Mohr-Coulomb yield function for Cosserat continua can be expressed in the same form as that for classical continua, that is,

$$f = \frac{1}{3} J_1 \sin\phi + \sqrt{J_2}(\cos\theta - \frac{1}{\sqrt{3}} \sin\theta \sin\phi) - c\cos\phi = 0, \quad (17) \qquad \theta = \frac{1}{3}\sin^{-1}(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}), \quad (18)$$

where c, ϕ and θ are cohesion, friction angle and Lade angle, respectively, and the three invariants are given in (10)~(12). If the associated flow rule is considered, the flow vector α can by written as:

$$\alpha^T = \frac{\partial f}{\partial \sigma} = C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3, \quad (19)$$

$$C_1 = \frac{\partial f}{\partial J_1}, \ C_2 = \frac{\partial f}{\partial \sqrt{J_2}}, \ C_3 = \frac{\partial f}{\partial J_3}, \quad (20) \qquad \alpha_1 = \frac{\partial J_1}{\partial \sigma}, \ \alpha_2 = \frac{\partial \sqrt{J_2}}{\partial \sigma}, \ \alpha_3 = \frac{\partial J_3}{\partial \sigma}. \quad (21)$$

For plane problems in Cosserat continua, α_1, α_2 and α_3 are

$$\alpha_1^T = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0\}, \quad (22)$$

$$\alpha_2^T = \frac{1}{2\sqrt{J_2}}\{s_{11}^{sym} \quad s_{22}^{sym} \quad s_{33}^{sym} \quad (s_{12}^{sym} + s_{12}^{ant}) \quad (s_{21}^{sym} + s_{21}^{ant}) \quad m_{23}/l \quad m_{13}/l\}, \quad (23)$$

$$\alpha_3^T = \left\{(s_{22}^{sym}s_{33}^{sym} + \frac{J_2^{sym}}{3}) \quad (s_{33}^{sym}s_{11}^{sym} + \frac{J_2^{sym}}{3}) \quad (s_{11}^{sym}s_{22}^{sym} - s_{12}^{sym}s_{21}^{sym} + \frac{J_2^{sym}}{3}) \quad -s_{33}^{sym}s_{21}^{sym} \quad -s_{33}^{sym}s_{12}^{sym} \quad 0 \quad 0\right\}. \quad (24)$$

Suppose that the forms of the other traditional yield functions (such as von Mises, Tresca and Drucker-Prager yield functions) for Cosserat continua can also be expressed in the same form as those for classical continua, then they can be established by adopting the modified stress invariants as well.

EXAMPLE

A biaxial test is modeled by finite element method with the proposed Mohr-Coulomb yield criterion for Cosserat continua, as shown in Fig.2. There is an imperfection at the bottom left corner. It can be seen that the width of the shearband is determined by the characteristic length l.

CONCLUSIONS

A Mohr-Coulomb yield criterion for Cosserat continua is established in detail by decomposing strain tensor and stress tensor into symmetric and skew-symmetric components, and modifying the formulae of stress invariants. The method to establish von Mises, Tresca and Drucker-Prager yield criteria for Cosserat continua are stated as well. The Cosserat theory can be employed to model more materials. The example of biaxial test demonstrates that the Cosserat model, which incorporates an internal length scale, is capable to resolve the mesh-dependence that exists in strain localization problem for classical continua

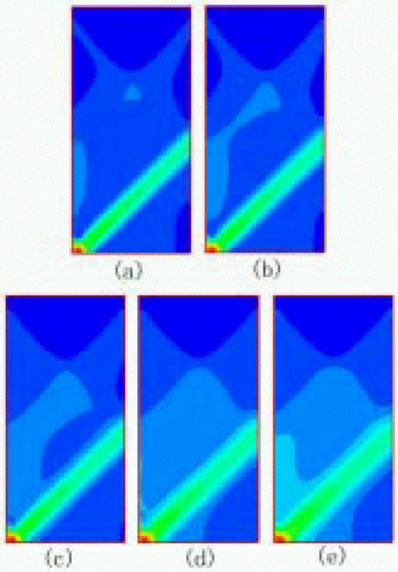

Fig 2 Contour of equivalent plastic strain

(a) $l = 0.001m$ (b) $l = 0.003m$

(c) $l = 0.005m$ (d) $l = 0.008m$

(e) $l = 0.01m$

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