THE LOCALIZATION OF PLASTIC STRAIN AND ROCK FRACTURE

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Summary Localization of strain and rock failure are considered from the aspects of dilatancial rigidly plastic and elasto-plastic models, based on the Coulomb-Mohr and Huber-Mises-Shleicher criterions [1,2]. For the plain strain and plain stress state, systems of differential equations are obtained for the velocity field. Conditions of their hyperbolicity are determined. Directions of localization and failure are identified with the characteristics of these systems.

PLAIN STRAIN

Let us consider rigidly plastic material, when elastic strains are fully neglected. This makes it possible to identify complete and plastic strains. Rock plasticity is associated with Coulomb-Mohr condition:

$$\max_n |\tau_n| + \sigma_n \tan\varphi = C\,, \tag{1}$$

where φ is the internal friction angle, C is the cohesion, τ_n and σ_n are tangential and normal stresses. Denominate α as coefficient of dilatancy, which characterizes change in the volume during shear along the limit planes. For the theories of plasticity, associated with the Coulomb-Mohr surface, $\alpha = \tan\varphi$. When α and φ are independent characteristics of the material, we have:

$$\begin{gathered}\tan 2\theta\,\frac{\partial v_x}{\partial x} - \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} - \tan 2\theta\,\frac{\partial v_z}{\partial z} = 0,\\ \left(a\cos 2\theta - b\right)\frac{\partial v_x}{\partial x} + \left(a\cos 2\theta + b\right)\frac{\partial v_z}{\partial x} = 0,\end{gathered} \tag{2}$$

where θ is the angle, which forms the first principal direction of the stress tensor σ_1 with the axis x. For the plain strain, $a = 1 + \alpha\tan\varphi,\ b = \alpha/\cos\varphi$. System of equations (2) has hyperbolic type at $-\tan(\pi/4 - \varphi/2) \le \alpha \le \tan(\pi/4 + \varphi/2)$ with the following characteristics:

$$\frac{dz}{dx} = \tan(\theta - \psi), \quad \frac{dz}{dx} = \tan(\theta + \psi), \quad \psi = \frac{\pi}{2} - \frac{1}{2}\arccos\frac{b}{a}\ , \tag{3}$$

where ψ is the angle formed by the characteristic line with direction σ_1.
Similarly, we consider rigid-plastic models, based on the Huber-Mises-Schleicher criterion:

$$T + \beta\sigma = k, \tag{4}$$

where T is the intensity of the tangential stress, β is the coefficient of internal friction, σ is the average normal stress. Take Λ as a coefficient of dilatancy, characterizing variation in the volume at the shear. For plastic models, associated with the Mises-Shleicher surface, $\Lambda = \beta$. On the basis of constitutive relations of the unassociated model [1] for the plain strain, we obtain system (2) as well, where should be taken as $a = \sqrt{1 - \Lambda^2/3}, \quad b = \Lambda$, and directions of the characteristics are determined by the formula (3). Condition of hyperbolicity of this system has the form $\Lambda \le \sqrt{3}/2$. As seen from (3), for the ideal of rigidly plastic body under plane strain when $\alpha = \Lambda = 0$, we have $\psi = 45°$. This result conforms to the classic theory of metal plasticity [3].

For the elastoplastic material, localization is associated with the loss of ellipticity and the first transition to the hyperbolicity of the equilibrium equations in terms of velocities. Since at the monotonous loading of the hardening material, plastic hardening modulus G_0 decreases, then the problem is reduced to the investigation into system of differential equations of the second order for the hyperbolicity and determination of the maximum of the hardening modulus depending on the direction of the characteristic. In this case, direction of the first characteristic and maximum hardening modulus G_0 of are determined [2]:

$$\cos 2\psi = -\frac{1}{2}\left(\frac{\alpha}{\cos\varphi + \alpha\sin\varphi} + \sin\varphi\right), \quad \frac{G_0}{\mu} = \frac{\cos\varphi + \alpha\sin\varphi}{8(1-\nu)}\cos\varphi\,(tg\varphi - \alpha)^2\,. \tag{5}$$

where μ is the elastic modulus of shear, ν is the Poison’s ratio.

PLAIN STRESS STATE

Consider plane stress state when using (4). Let axis y coincide with the second principal direction of the stress tensor, then for $\sigma_2 = 0$ criterion (4) in the principal axes of stress has the form:

$$\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2 = \left(\sqrt{3}k - \beta\,\frac{\sigma_1 + \sigma_3}{\sqrt{3}}\right)^2. \tag{6}$$

Type of the curve (6) in the plane σ_1, σ_3 is determined by the value of β. If $\beta < \sqrt{3}/2$, then (6) is the ellipse. When $\beta = \sqrt{3}/2$, it is a parabola; when $\beta > \sqrt{3}/2$, it is a hyperbola. Determination of the velocity field on the basis of (6) leads to the system of differential equations (2), in which $a = \sin\omega$, $b = 2\Lambda/3 + \cos\omega/\sqrt{3}$, where ω is the angle of the stress state type [3]. For the uniaxial tension and compression, respectively $\omega = \pi/3$, $\omega = 2\pi/3$. Condition of hiperbolicity for the system (2) has the form: $12\cos^2\omega + 4\sqrt{3}\Lambda\cos\omega + 4\Lambda^2 - 9 \le 0$. Direction of the characteristic is determined from (3):

$$\psi = \frac{\pi}{2} - \frac{1}{2}\arccos\left(\frac{\mathrm{ctg}\,\omega}{\sqrt{3}} + \frac{2\Lambda}{3\sin\omega}\right). \tag{7}$$

For the stress states, corresponding to the verges of the Coulomb-Mohr pyramid, determination of the velocity field characteristic reduces to the plane strain or to the known solutions for the ductile metals [3]. For the uniaxial tension and compression, corresponding to the edges of the pyramid. Directions of the corresponding characteristics ψ_t and ψ_c are determined by the following formulae:

$$\cos 2\psi_t = -\frac{\cos\varphi + \alpha(3 + \sin\varphi)}{3\cos\varphi + \alpha(1 + 3\sin\varphi)}, \quad \cos 2\psi_c = \frac{\cos\varphi - \alpha(3 - \sin\varphi)}{3\cos\varphi - \alpha(1 - 3\sin\varphi)}. \tag{8}$$

As seen from (7) and (8), for the rigidly plastic body under plane stress when $\alpha = \Lambda = 0$, we have for uniaxial tension $\psi_t \approx 54.7°$ and compression $\psi_c \approx 35.3°$. This result conforms to the classic theory of metal plasticity [3].

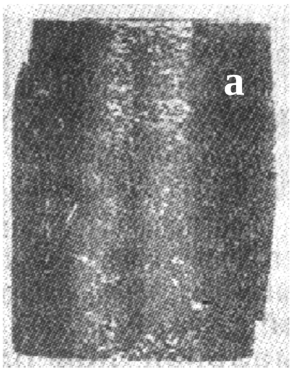
For the hardening elastoplastic metal, when $\alpha = \varphi = 0$, maximum value of the modulus G_0 during tension is reached when $\psi_t \approx 50°$ and under compression $\psi_c \approx 40°$. In both cases, localization of plastic strain at the positive modulus of hardening G_0 [2]:

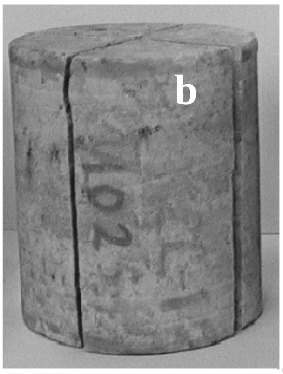
$$G_0 = \frac{\mu(1+\nu)}{4(12-7\nu)},$$

that better reveals behavior of the real materials, than results in [1].

Comparison of calculation results by criterions (1) and (6) with Coffin's experimental data on the grey cast iron [4] showed their correspondence. From these experiments for the uniaxial compression, we obtained $\beta = 0.62$, $\varphi = 20°$. If for the uniaxial compression in (7) we take $\Lambda = \beta$, then obtain $\psi_c \approx 49°$. If we take $\alpha = \tan\varphi$ in (8) then $\psi_c = 45°$. These results coincide with the experimental data of S.I. Gubkin [5], where cylindrical grey cast iron samples were failed at the angle of 45° (Fig. a).

It is well-known [4] that if sample edges under compression are in direct contact with platens of the testing machine and edges are lubricated, then the sample is failed by "splitting" along the axial plane, when $\psi_c = 90°$. In Fig. b, results are presented for the experiments on the limestone. This result follows from (7), if we take $\Lambda = \sqrt{3}$ and from (8), if we take $\alpha = \tan(\pi/4 + \varphi/2)$. Application of this approach gives coincidence of the theoretical and experimental results.





Conclusion

The analysis is performed for two rigidly plastic and elastoplastic models based on the Coulomb – Mohr and Huber – Mises – Shleiher criteria. The states of plane deformation and plain stress are considered; the characteristics of the velocity fields in the sphere of hyperbolic equations are determined for them. If the directions of failure in brittle metals and rocks are equated with the velocity field characteristics, then basing on considered models for (1) and (4) criteria, it is possible to determine not only the stresses at which the failure begins, but also the angles at which the failure take place.

References

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