THE MATERIAL POINT METHOD IN SOIL MECHANICS PROBLEMS

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Summary The material point method – the finite element method formulated in an arbitrary Lagrangian–Eulerian description of motion – is applied to large strain problems of soil mechanics. Two-dimensional, dynamic problems of pile driving (axisymmetry) and retaining wall failure (plane strain) are analysed. The case of non-cohesive models of the soil is analysed – the elastic–plastic and elastic–viscoplastic material models are applied.

INTRODUCTION

The problems of large strains are still hard to analyse despite of existence of many well-developed computational techniques. This statement refers to some problems of soil mechanics, e.g., the pile driving or landslide problems. For example, the finite element method, the tool used most frequently in engineering analyses, is not sufficiently robust in the case of such problems when formulated in the Lagrangian description of motion. The excessive distortions of an element mesh deforming together with an analysed body lead to inaccuracies in the solution approximation or even to failure of a calculation process due to negative values of Jacobian determinants at points of numerical integration. The use of re-meshing techniques is not a sufficient measure because it is time-consuming and introduces additional errors due to the projection of the solution from a deformed mesh to a regenerated one.

Recently, two groups of computational methods handling the problems of large strains have been developed intensely: the – so called – point based or meshless methods and the methods formulated in an arbitrary Lagrangian–Eulerian description of motion. The material point method (MPM), used as a tool of analysis in the present paper, can be classified to both the groups due to its features. The method, well-known in fluid mechanics as a particle-in-cell method, was introduced by Harlow [1] in 1964 and adapted to problems of solid mechanics by Burgess *et al.* [2] and Sulsky *et al.* [3, 4] about ten years ago. Application of the method to problems of granular flow in a silo has been described in [5].

In the present paper, two-dimensional problems of pile driving and failure of a retaining wall are investigated.

SETTING OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^3$ denote a region occupied by the granular body at instant $t \in I \equiv [0,T]$, where $T > 0$. Let us assume that the boundary of the body consists of two parts Γ_u and Γ_σ such that $\overline{\Gamma_u} \cup \overline{\Gamma_\sigma} = \partial\Omega$ and $\Gamma_u \cap \Gamma_\sigma = \emptyset$.

The solution of the dynamic problem satisfies the equation of virtual work:

$$\int_\Omega (\varrho\, a_i\, w_i + \sigma_{ij} w_{i,j})\, \mathrm{d}x = \int_\Omega \varrho\, b_i\, w_i\, \mathrm{d}x + \int_{\Gamma_\sigma} t_i\, w_i\, \mathrm{d}s \quad \forall \boldsymbol{w} \in V_0, \tag{1}$$

where V_0 denotes the space of kinematically admissible fields of displacements, σ_{ij} the Cauchy stress tensor, ϱ mass density, b_i and a_i are the vectors of mass forces and acceleration, respectively, t_i denotes the Cauchy stress vector given on Γ_σ.

The displacement, velocity and stress fields satisfy the following initial conditions: $u_i(0) = u_i^0$, $\dot{u}_i(0) = 0$, $\sigma_{ij}(0) = \sigma_{ij}^0$, where u_i^0 and σ_{ij}^0 are the initial fields of displacements and stresses, respectively.

The case of non-cohesive soil is investigated in the paper. Two constitutive models of the soil are considered: the elastic–perfectly plastic and elastic–viscoplastic ones. In both the models, the Drucker–Prager yield condition and a non-associative flow rule are involved.

Let f denote the yield function, $f(\sigma_{ij}) = q - mp$, where $m = 18 \sin\varphi/(9 - \sin^2\varphi)$ is a function of the angle of internal friction, φ, p and q are invariants of the stress tensor, $p = -\sigma_{ii}/3$, $q = \sqrt{3/2\, s_{ij} s_{ij}}$, where $s_{ij} = \sigma_{ij} + p\,\delta_{ij}$ denotes the deviatoric part of the stress tensor. The constitutive relations for the elastic–perfectly plastic material model are as follows:

$$\dot{p} = K\, d_{kk}, \qquad e_{ij} = e_{ij}^{\mathrm{e}} + e_{ij}^{\mathrm{p}}, \qquad e_{ij}^{\mathrm{e}} = \frac{1}{2\,G} \overset{\triangledown}{s}_{ij}, \qquad e_{ij}^{\mathrm{p}} = \begin{cases} \dot{\lambda}\, \dfrac{\partial g}{\partial s_{ij}} & \text{if } f(\sigma_{ij}) = 0, \\ 0 & \text{if } f(\sigma_{ij}) < 0, \end{cases} \tag{2}$$

where $\dot{\lambda} \geq 0$, g denotes the plastic potential defined by the relation $g = q$. The following notation is used above: $d_{ij} = (v_{i,j} + v_{j,i})/2$ is the rate-of-deformation tensor, e_{ij}^{e} and e_{ij}^{p} are parts of its deviator, $e_{ij} = d_{ij} - d_{kk}\,\delta_{ij}/3$, the elastic and plastic ones, respectively, $\overset{\triangledown}{\sigma}_{ij} = \dot{\sigma}_{ij} - \sigma_{ik}\,\omega_{kj} - \sigma_{jk}\,\omega_{ki}$ is the Jaumann rate of the stress tensor, $\omega_{ij} = (v_{j,i} - v_{i,j})/2$ the spin, K and G are the bulk and shear moduli, respectively.

The second material model is a viscoplastic regularisation of the model described above. The same yield condition and plastic potential are employed. To define the viscoplastic constitutive relations, the second and fourth equations in (2) are replaced by

$$e_{ij} = e^{\mathrm{e}}_{ij} + e^{\mathrm{vp}}_{ij}, \quad e^{\mathrm{vp}}_{ij} = \gamma \langle \Phi(f) \rangle \frac{\partial g}{\partial \sigma_{ij}} \quad \text{with} \;\; \Phi\left(f(\sigma_{ij})\right) = \left(\frac{q - m\,p}{m\,p}\right)^N, \;\; \langle \Phi(f) \rangle = \begin{cases} \Phi(f) & \text{if } f > 0 \\ 0 & \text{if } f \leq 0 \end{cases}, \quad N > 0.$$

THE MATERIAL POINT METHOD

Two kinds of space discretisation are utilised in the material point method. The Lagrangian discretisation is done by dividing the region occupied initially by the analysed body into a set of subregions – each of them represented by one its points called a material point. The mass density field is expressed as follows: $\varrho(\boldsymbol{x}) = \sum_{P=1}^{N} M_P \, \delta(\boldsymbol{x} - \boldsymbol{X}_P)$, where M_P and $\boldsymbol{X}_P$ denote the mass and the position of the P-th material point, $\delta(\boldsymbol{x})$ is the Dirac δ-function. Another kind of space discretisation is related to an Eulerian finite element mesh, called a computational mesh, covering the virtual position of the analysed body. This mesh can be changed arbitrarily during calculations or remain constant. After substituting the δ-function representation of the mass density field to the equation of virtual work (1) and expressing the field of acceleration, a_i, and the weight functions, w_i, by the shape functions and nodal parameters, defined on the computational mesh as in the finite element method, we obtain the following system of dynamic equations:

$$\mathbf{M}\,\mathbf{a} = \mathbf{F} - \mathbf{R}, \tag{3}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{a}$ the vector of nodal accelerations, $\mathbf{F}$ and $\mathbf{R}$ are the vectors of external and internal nodal forces, respectively. The main difference between the finite element (FEM) and material point (MPM) methods is based on the fact that the state variables are traced at the material points, defined independently of the computational mesh in MPM, and at integration points connected with elements in FEM.

EXAMPLES

Two examples of application of MPM to soil mechanics are presented: the plane problem of failure of a retaining wall in Fig. 1 and the axisymmetric problem of pile driving with constant velocity in Fig. 2. Several stages of deformation processes are shown in the figures. The viscoplastic and elastic material models have been used in the calculations for soil (sand) and walls and pile, respectively. The dynamic equations (3) have been solved by the use of the explicit time integration scheme.

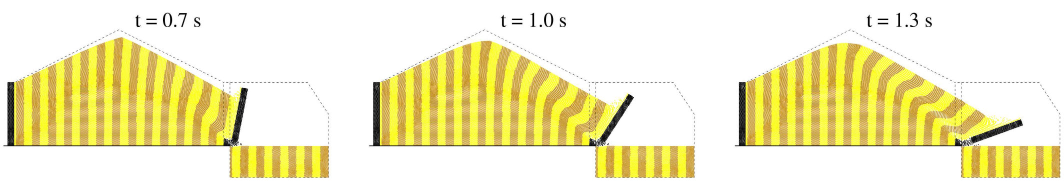

Figure 1: Problem of retaining wall failure

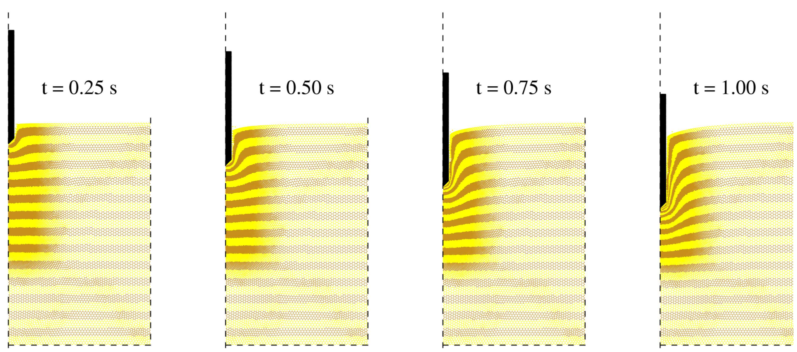

Figure 2: Problem of pile driving

References

[1] Harlow F.H.: The particle-in-cell computing method for fluid dynamics. In Adler B., Fernbach S., Rotenberg M., eds., Methods for Computational Physics, **Vol. 3**:319–343, Academic Press, NY 1964.

[2] Burgess D., Sulsky D., Brackbill J.U.: Mass matrix formulation of the FLIP particle-in-cell method. *J. Computat. Phys.* **103**:1–15, 1992.

[3] Sulsky D., Chen Z., Schreyer H.L.: A particle method for history-dependent materials. *Comput. Meth. Appl. Mech. Engrg.* **118**:179–196, 1994.

[4] Sulsky D., Schreyer H.L.: Axisymmetric form of the material point method with applications to upsetting and Taylor impact problems. *Comput. Meth. Appl. Mech. Engrg.* **139**:409–429, 1996.

[5] Więckowski Z., Youn S.K., Yeon J.H.: A particle-in-cell solution to the silo discharging problem. *Int. J. Numer. Meth. Eng.* **45**:1203–1225, 1999.