MESHLESS LBIE FORMULATIONS FOR VISCOELASTIC THIN PLATES

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Summary In this paper, simply supported and clamped thin viscoelastic plates are analyzed. Linear viscoelasticity has been considered for which the correspondence principle is applied. The biharmonic differential equation for Laplace transform of deflections is decomposed into two Poisson equations. Local boundary integral equations are derived for this system of equations. The meshless approximation based on the moving least-squares is employed for the implementation.

1. INTRODUCTION

A viscoelastic problem can be solved either in the time domain directly or in the Laplace-transformed domain with the inversion being carried out later for a given time. The direct approach is leading to integro-differential field equations. In the approach based on the Laplace transform technique the correspondence principle [1] is utilized. The later approach is simpler and computationally more efficient if an accurate Laplace inversion technique is available. This method has been successfully applied to viscoelastic plate analyses by the boundary element method (BEM) [2].

If the correspondence principle is applied the governing partial differential equation is of the forth order. In case of plates with clamped edges and/or simply supported straight edges, it is possible to decompose the governing equation into a system of two Poisson equations [3] for Laplace transforms of deflection and its Laplacian, respectively. The meshless approach based on local boundary integral equations (LBIE) is applied to elastic plates [3]. In this paper the idea is extended to viscoelastic plates. Nodes are randomly spread on the middle plane of the plate. Each node is surrounded by a circle for which LBIE can be formulated.

2. LOCAL BOUNDARY INTEGRAL EQUATIONS

In accordance with Boltzmann´s principle the linear viscoelastic model take the form

$$\sigma_{ij}(\mathbf{x},t) = c_{ijkl}(t)\varepsilon_{kl}(\mathbf{x},t) + \int_0^t J_{ijkl}(t-\tau)\varepsilon_{kl}(\mathbf{x},\tau)d\tau \quad , \tag{1}$$

where σ_{ij} , ε_{ij} are the stress and small strain tensors, respectively, and J_{ijkl} the relaxation moduli. The first term in (1) corresponds to the elastic deformations. Viscoelastic materials are often simulated by a rheological model. In such a case the constitutive equations for a linear isotropic viscoelastic material is given as

$$\hat{\mathbf{D}}_1 s_{ij} = \hat{\mathbf{F}}_1 e_{ij} \quad , \qquad \hat{\mathbf{D}}_2 \sigma_{kk} = \hat{\mathbf{F}}_2 \varepsilon_{kk} \quad , \tag{2}$$

where $\hat{\mathbf{D}}_a$ and $\hat{\mathbf{F}}_a$ are the linear differential operators, s_{ij} and e_{ij} are deviators of stress and strain, respectively.

The Laplace transform of the convolution product is given by a simple product of Laplace transforms. Then, one can write

$$\bar{\sigma}_{ij} = 2\bar{\mu}\,\bar{\varepsilon}_{ij} + \bar{\lambda}\delta_{ij}\,\bar{\varepsilon}_{kk} \tag{3}$$

or

$$\bar{s}_{ij} = 2\bar{\mu}\,\bar{e}_{ij}$$

$$\bar{\sigma}_{kk} = 3\bar{K}\,\bar{\varepsilon}_{kk} \quad , \qquad \bar{K} = \left(2\bar{\mu} + 3\bar{\lambda}\right)/3 \quad , \tag{4}$$

where $\bar{\mu}$, $\bar{\lambda}$ and $\bar{K}$ are Laplace transforms of Lame moduli and bulk modulus, respectively.

In the case of quasi-static viscoelasticity the governing equation for bending of thin elastic plates in terms of the Laplace transforms become

$$\nabla^2\nabla^2\,\bar{w}(\mathbf{x},p) = \frac{\bar{q}(\mathbf{x},p)}{\bar{D}(p)} \quad , \tag{5}$$

where $\bar{D}(p)$ is the plate rigidity in the transformed domain given by

$$\bar{D}(p) = \frac{\bar{E}\,h^3}{12\left(1-\bar{\nu}^2\right)} \quad ,$$

with plate thickness *h*, Young modulus $\bar{E}(p)$ and Poisson ratio $\bar{\nu}(p)$. Laplace transform parameter is denoted by *p* and $\bar{q}$ is a load intensity in Laplace transform domain.

The governing equation (5) is decomposed into two Poisson equations:

$$-\bar{D}\nabla^2\bar{w}(\mathbf{x},p) = \bar{m}(\mathbf{x},p) \tag{6}$$

$$\nabla^2 \overline{m}(\mathbf{x}, p) = -\overline{q}(\mathbf{x}, p) \quad . \tag{7}$$

The Laplacian of deflection, $\nabla^2 \overline{w}(\mathbf{x}, p)$, has no physical meaning inside the plate. On a clamped edge, however $\overline{m}(\mathbf{x}, p)$ is equal to the bending moment. On a simply supported edge this is only the case if the edge is straight.
The solutions for the Poisson equations (6) and (7) can be expressed in integral form as

$$\overline{m}(\mathbf{y}, p) = \int_{\Omega} \overline{q}(\mathbf{x}, p) U(\mathbf{x}-\mathbf{y}) d\Omega + \int_{\Gamma} \left[\frac{\partial \overline{m}}{\partial n}(\mathbf{x}, p) U(\mathbf{x}-\mathbf{y}) - \overline{m}(\mathbf{x}, p) \frac{\partial U}{\partial n}(\mathbf{x},\mathbf{y}) \right] d\Gamma \tag{8}$$

$$\overline{w}(\mathbf{y}, p) = \frac{1}{\overline{D}(p)} \int_{\Omega} \overline{m}(\mathbf{x}, p) U(\mathbf{x}-\mathbf{y}) d\Omega + \int_{\Gamma} \left[\frac{\partial \overline{w}}{\partial n}(\mathbf{x}, p) U(\mathbf{x}-\mathbf{y}) - \overline{w}(\mathbf{x}, p) \frac{\partial U}{\partial n}(\mathbf{x},\mathbf{y}) \right] d\Gamma \quad , \tag{9}$$

where $\mathbf{y} \in \Omega$ and the kernel $U = (1/2\pi) \ln 1/r$ is the fundamental solution of the Laplace equation for an infinite plane.
The global integral equations (8) and (9) are coupled. Thus, they cannot be solved separately. Domain discretization of the coupled system and integration over the entire domain is not convenient for practical implementation. The approach based on a domain decomposition seems to be more convenient. The analyzed domain can be covered by a set of simple, regular subdomains. Integral equations (8) and (9) can be directly applied to each subdomain. Because of regular shape of the subdomains, all integrals can be evaluated easily. Nodal points are distributed not only on the boundary but also inside the investigated domain. No mesh is generated for the approximation of the field variables. The integral equations are considered for small subdomains Ω_s . Hence, no boundary densities (Laplace transforms) are prescribed on local boundary $\partial\Omega_s$ as long as it lies entirely inside Ω . To reduce number of unknowns on $\partial\Omega_s$ the modified fundamental solution [3]

$$U^*(r) = \frac{1}{2\pi} \ln \frac{r_0}{r} \tag{10}$$

is used in local boundary integral equations (LBIEs) for circular subdomains with radius r_0 . The modified fundamental solution is the Green function (for the Laplace equation) vanishing on the boundary of the circular subdomain ($r = |\mathbf{x}-\mathbf{y}| = r_0$) .
This leads to the following simplifications of LBIEs:

$$\overline{m}(\mathbf{y}, p) = \int_{\Omega_s} \overline{q}(\mathbf{x}, p) U^*(\mathbf{x}-\mathbf{y}) d\Omega - \int_{\partial\Omega_s} \overline{m}(\mathbf{x}, p) \frac{\partial U^*}{\partial n}(\mathbf{x},\mathbf{y}) d\Gamma \tag{11}$$

$$\overline{w}(\mathbf{y}, p) = \frac{1}{\overline{D}(p)} \int_{\Omega} \overline{m}(\mathbf{x}, p) U^*(\mathbf{x}-\mathbf{y}) d\Omega - \int_{\partial\Omega_s} \overline{w}(\mathbf{x}, p) \frac{\partial U^*}{\partial n}(\mathbf{x},\mathbf{y}) d\Gamma \quad . \tag{12}$$

The moving least-square (MLS) method is considered as the domain type approximation of the Laplace transforms in the LBIEs (11)-(12). The approximated function can be written as [4]

$$u^h(\mathbf{x}) = \sum_{a=1}^{n} \phi^a(\mathbf{x}) \hat{u}^a \quad . \tag{13}$$

Making use of the MLS approximation (13) for $\overline{m}(\mathbf{x}, p)$ and $\overline{w}(\mathbf{x}, p)$, the LBIEs yield the set of algebraic equations for $\hat{m}^a$ and $\hat{w}^a$. The time-dependent values of any of the transformed variables are obtained by an inverse transform. We have used Stehfest´s inversion technique.

References

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