Three-Dimensional Analytical and Semi-Analytical Investigation of Free-Corner Effects

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Summary Analytical and semi-analytical approaches to 3D laminate corner effects are presented. First, we use an analytical approach in the form of a layerwise assumed-stress approach for arbitrary lamination schemes. Second, a hybrid layerwise displacement-based variational formulation proves to be highly efficient for cross-ply plates. As a closure, a novel semi-analytical approach to the 3D asymptotic corner behaviour using the boundary finite element method is presented.

STRESS CONCENTRATIONS IN LAMINATES: FREE-EDGE AND FREE-CORNER EFFECTS

Stress fields at the interfaces of dissimilar layers near the free edges of laminates (see Fig. 1) in general are of a localized 3D singular nature (so-called free-edge effects, [1]). Such effects rapidly decay at some distance from the edges and are evoked by the discontinuous change of the material properties in the interfaces, in the inner laminate regions classical laminate plate theory (CLPT) is recovered. Free-edge effects are well understood [1], however the local behaviour of layered structures in the vicinity of free corners (so-called free-corner effect) has gone nearly unnoticed in the open literature. A profound understanding of such stress concentration phenomena which require highly refined analysis techniques is of vital practical importance since interlaminar stresses may be the cause of corresponding interlaminar failure modes such as delaminations. Since purely numerical studies of free-corner effects usually require vast computational effort, it is of particular interest to develop approximate analytical and semi-analytical methods for the accurate computation of free-corner displacements, strains and stresses at reasonable computational expenses.

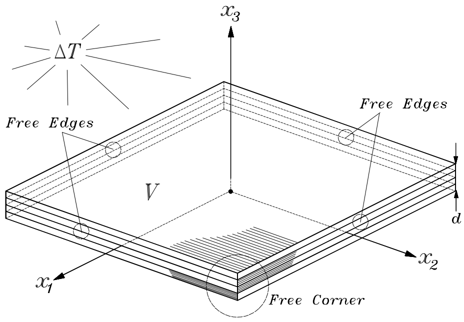

***Fig. 1:** Laminate under thermal load.*

3D ANALYTICAL AND SEMI-ANALYTICAL APPROACHES TO FREE-CORNER EFFECTS

We present two analytical methods for symmetric rectangular cross-ply laminates made of n plies under uniform thermal load ΔT. The approaches are sorted as i) a layerwise assumed-stress approach and ii) a hybrid layerwise displacement-based variational formulation. The first approach i) can be expanded on arbitrary laminate layups. As a closure we employ the semi-analytical boundary finite element method (BFEM) for the asymptotic eigenanalysis of free-corner effects in laminates of arbitrary layups and with varying corner opening angles beyond the rectangular case.

Layerwise assumed-stress approach for cross-ply laminates under thermal load

This method [2] uses a variable-separable formulation with simple exponential and polynomial terms for the inplane normal components σ_{11}, σ_{22} of Cauchy's stress tensor (i.e. $\sigma_{12}=0$) in the k-th layer and a priori fulfills all traction-free boundary conditions. Integrated 3D equilibrium conditions yield closed-form expressions for all stress components

$$\sigma_{11}^{(k)}=\left(1-\left(1+\varphi_1 x_1\right)e^{-\varphi_1 x_1}\right)\left(1+\varphi_2 e^{-\varphi_3 x_2}\right)\left(A_1^{(k)}x_3+A_2^{(k)}\right),\quad \sigma_{22}^{(k)}=\left(1+\varphi_4 e^{-\varphi_5 x_1}\right)\left(1-\left(1+\varphi_6 x_2\right)e^{-\varphi_6 x_2}\right)\left(A_3^{(k)}x_3+A_4^{(k)}\right),$$

$$\sigma_{13}^{(k)}=-\varphi_1^2 x_1 e^{-\varphi_1 x_1}\left(1+\varphi_2 e^{-\varphi_3 x_2}\right)\left(\tfrac{1}{2}A_1^{(k)}x_3^2+A_2^{(k)}x_3+B_1^{(k)}\right),\quad \sigma_{23}^{(k)}=-\varphi_6^2 x_2 e^{-\varphi_6 x_2}\left(1+\varphi_4 e^{-\varphi_5 x_1}\right)\left(\tfrac{1}{2}A_3^{(k)}x_3^2+A_4^{(k)}x_3+B_3^{(k)}\right),$$

$$\begin{aligned}\sigma_{33}^{(k)}&=\varphi_1^2\left(1-\varphi_1 x_1\right)e^{-\varphi_1 x_1}\left(1+\varphi_2 e^{-\varphi_3 x_2}\right)\left(\tfrac{1}{6}A_1^{(k)}x_3^3+\tfrac{1}{2}A_2^{(k)}x_3^2+B_1^{(k)}x_3+B_2^{(k)}\right)\\&\quad+\varphi_6^2\left(1-\varphi_6 x_2\right)e^{-\varphi_6 x_2}\left(1+\varphi_4 e^{-\varphi_5 x_1}\right)\left(\tfrac{1}{6}A_3^{(k)}x_3^3+\tfrac{1}{2}A_4^{(k)}x_3^2+B_3^{(k)}x_3+B_4^{(k)}\right),\end{aligned}\qquad (1)$$

wherein the layerwise constants A_i and B_i (i=1,...,4) are determined from the requirements of recovery of CLPT in the inner laminate regions and of interlaminar stress continuity and traction-free laminate facings, respectively. The unknown constants φ_j (j=1,...,6) describe the decaying rates of the free-corner perturbations and are determined iteratively from the principle of minimum complementary energy which reads in a contracted vector-matrix notation with the layerwise compliance matrix $\underline{S}$ and the stresses and thermal expansion coefficients arranged in the arrays $\underline{\sigma}$ and $\underline{\alpha}_t$:

$$\overline{\Pi}=\tfrac{1}{2}\sum_{j=1}^{j=n}\iiint_{V^{(j)}}\underline{\sigma}^{(j)^T}\,\underline{S}^{(j)}\,\underline{\sigma}^{(j)}\mathrm{d}V^{(j)}+\sum_{j=1}^{j=n}\iiint_{V^{(j)}}\underline{\sigma}^{(j)^T}\,\underline{\alpha}_t^{(j)}\Delta T^{(j)}\mathrm{d}V^{(j)}\overset{!}{=}\text{Min.}\qquad (2)$$

The computational effort needed for this method is uncoupled from the number of layers in the laminate. The approach yields quite accurate stress results and can be easily extended for the calculation of arbitrary layups by using adequate additional layerwise assumptions for the inplane shear stress σ_{12} .

Hybrid layerwise displacement-based variational formulation for cross-ply laminates under thermal load

This approach is based on the discretization of the laminate into a number n_L of mathematical layers through the thickness and employs a C^0-continuous layerwise higher-order displacement formulation with the layerwise CLPT terms u_{i0} , unknown inplane functions $U_1(x_1)$, $U_2(x_2)$, $U_{31}(x_1)$, $U_{32}(x_2)$ in the k-th layer defined in the lower interface k and the upper interface k+1, and a priori assumed Lagrangian thickness interpolants $\psi_1(x_3)$, $\psi_2(x_3)$ ($\alpha=1,2$):

$$u_\alpha^{(k)} = u_{\alpha 0}^{(k)} + U_\alpha^{(k)}(x_\alpha)\psi_1^{(k)}(x_3) + U_\alpha^{(k+1)}(x_\alpha)\psi_2^{(k)}(x_3),$$
$$u_3^{(k)} = u_{30}^{(k)} + \left(U_{31}^{(k)}(x_1) + U_{32}^{(k)}(x_2)\right)\psi_1^{(k)}(x_3) + \left(U_{31}^{(k+1)}(x_1) + U_{32}^{(k+1)}(x_2)\right)\psi_2^{(k)}(x_3). \tag{3}$$

The principle of minimum potential (with the layerwise strains arranged in the array $\underline{\varepsilon}$, and the stiffness matrix $\underline{C}$)

$$\Pi = \tfrac{1}{2}\sum_{j=1}^{j=n_L} \iiint_{V^{(j)}} \underline{\varepsilon}^{(j)^T} \underline{C}^{(j)} \underline{\varepsilon}^{(j)} \mathrm{d}V^{(j)} - \sum_{j=1}^{j=n_L} \iiint_{V^{(j)}} \underline{\sigma}^{(j)^T} \underline{\alpha}_t^{(j)} \Delta T^{(j)} \mathrm{d}V^{(j)} + \tfrac{1}{2}\sum_{j=1}^{j=n_L} \iiint_{V^{(j)}} \underline{\alpha}_t^{(j)^T} \Delta T^{(j)} \underline{C}^{(j)} \underline{\alpha}_t^{(j)} \Delta T^{(j)} \mathrm{d}V^{(j)} \overset{!}{=} \mathrm{Min.} \tag{4}$$

yields a set of coupled ordinary Euler-Lagrange differential equations decomposed with respect to the two inplane coordinates which allow for a closed-form solution. The boundary conditions of traction free edges are fulfilled in an integral sense. As this approach applies a discretization through the thickness yet allows a closed-form solution for all state variables within the layer plane, it is appropriate to speak of a hybrid analysis method. The approach is in excellent agreement with comparative finite element calculations (see Fig. 2), however it requires the numerical solution of an involved eigenproblem which makes the computational effort a function of the number of layers.

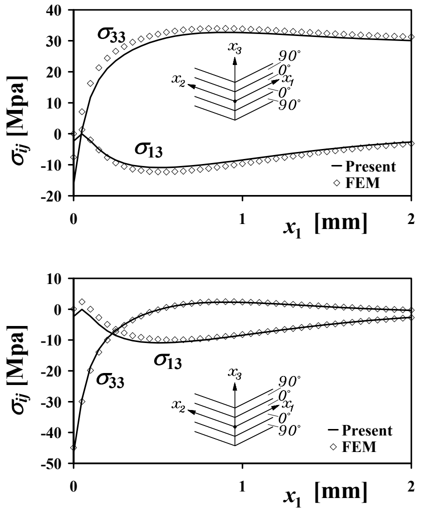

Fig. 2: Interlaminar stresses σ_{13} *and* σ_{33} *alongside the free edge* x_1, x_2=0 (top) *and* x_1, x_2=2 mm (bottom) *directly below the* 90°/0°*-interface of a* [90°/0°]$_S$*-layup* with d=2.0 mm.

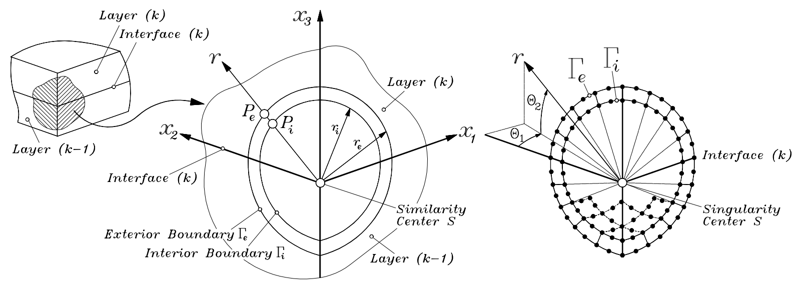

Fig. 3: Principle of scalability, discretization of an interface corner according to the boundary finite element method.

Asymptotic eigenanalysis by the boundary finite element method

Asymptotic analyses of 3D corner stress singularities for arbitrary layups and corner angles (see Fig. 3) are performed using the boundary finite element method (BFEM) [3]. This novel approach assumes scalability / similarity of the structure with respect to a similarity center S where the radial coordinate r is located and requires the discretization of a finite element cell between two similar arbitrary boundaries Γ_i, Γ_e only. Conditions of equilibrium and similarity relations lead to an eigenproblem

$$\underline{H}\underline{\Phi} = \underline{\Phi}\underline{\lambda} \quad \text{with} \quad \underline{\Phi} = \begin{bmatrix} \underline{\Phi}_{11} & \underline{\Phi}_{12} \\ \underline{\Phi}_{21} & \underline{\Phi}_{22} \end{bmatrix}, \; \underline{\Lambda} = \begin{bmatrix} \underline{\lambda} & \underline{0} \\ \underline{0} & -\underline{\lambda} \end{bmatrix}, \tag{5}$$

and to a simple system of differential equations for the nodal displacements $\underline{u}$ that allow for a closed-form solution as

$$\underline{u} = \underline{\Phi}_{11}\left[\delta_{ij}\left(\tfrac{r}{r_0}\right)^{\lambda_i - \frac{1}{2}}\right]\underline{\Phi}_{11}^{-1}\underline{u}_0. \tag{6}$$

The Hamiltonian matrix $\underline{H}$ as well as the problem eigenvalues and eigenvectors arranged in the matrices $\underline{\Lambda}$ and $\underline{\Phi}$ solely depend on the geometry and material data of the finite element cell. In the circumferential directions the method converges in the FEM sense and is thus of a semi-analytical nature. In essence, the BFEM can be described as a fundamental-solution-less boundary element formulation based on finite elements accounting for anisotropic material behaviour without complications. Hence, advantages of both the FEM and the BEM are unified in the BFEM approach. The BFEM is particularly effective for the computation of the order and eigenmodes of stress singularities [3] due to the exponential structure of the solution. Strains and stresses can be computed straightforwardly from the displacements.

CONCLUSIONS

Two closed-form analytical approaches to free-corner effects are presented, both of which have their specific advantages and disadvantages, corresponding to the needed computational effort and available degree of accuracy. Extensive results on free-corner stress fields are reported for the first time, as well as the application of the semi-analytical boundary finite element method to the asymptotic analysis of free corners of laminates with arbitrary layups and corner opening angles.

References

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[2] Mittelstedt C., Becker W.: Interlaminar Stress Concentrations in Layered Structures, Part II: Closed-Form Analysis of Stresses at Laminated Rectangular Wedges with Arbitrary Non-Orthotropic Layup. *J. Compos Mater*, 38(12):1063-1090, 2004.

[3] Wolf J.P.: The Scaled Boundary Finite Element Method. John Wiley & Sons, Chichester et al. 2003.