Strongly Coupled Inversion of Rayleigh Dispersion and Attenuation Curves

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Summary This memoir illustrates an elegant technique for the solution of the Rayleigh inverse problem in arbitrarily dissipative, linear viscoelastic media. The technique is based on using the holomorphic properties of Rayleigh phase velocity viewed as an analytic mapping of the speed of propagation of transversal waves, and a recently obtained result on the solution of the Rayleigh forward problem.

INTRODUCTION

In the seismological literature the term simultaneous inversion of surface wave data is used to define a procedure for determining the phase velocity and quality factor profiles $V_S(y)$ and $Q_S(y)$ of transversal waves in a dissipative, layered medium from the inversion of experimentally measured surface wave dispersion and attenuation curves. In most cases the inversion procedure is based on the assumption of *weak dissipation* which allows to solve the problem using little more computational efforts than those required for the solution of the corresponding elastic problem.
Dissipative media are inherently dispersive and as a result the speed of propagation and the attenuation of body waves are not independent parameters. Inversion of surface wave data by accounting for material dispersion is preferable since this procedure will satisfy the physical *principle of causality*. However a weakly coupled inversion is causal only for values of the quality factor $Q > 10$ (Lai & Rix, 2002). For smaller values the inversion is causal only approximately with a degree of approximation that deteriorate as Q gets smaller (i.e. as the medium become more dissipative). Even though the assumption of weak dissipation is often appropriate to the values of Q exhibited by geomaterials subjected to seismic excitations, there are situations of practical relevance where this is no longer true. Furthermore, besides the applications, it is of interest to examine the consequences of relaxing the assumption of weak dissipation in studying the propagation of surface waves.
The homogeneous boundary value problem associated with the propagation of surface waves in arbitrarily dissipative, linear viscoelastic media can be formally represented by a complex-valued, differential eigenproblem, whose rigorous solution can be obtained using a using some well known results of complex-variable theory, in particular the Cauchy residue theorem and its corollaries (Lai & Rix, 2002). This result applied to Rayleigh waves has been profitably used to develop a strongly coupled inversion algorithm for determining the $V_S(y)$ and $Q_S(y)$ profiles of a general, linear viscoelastic, layered systems. The capabilities of the algorithm are illustrated through a numerical simulation.

WAVE PROPAGATION IN LINEAR VISCOELASTIC CONTINUA

A complete description of an arbitrarily dissipative, linear viscoelastic isotropic solid requires two material functions, which may be specified in either the time or frequency domain. The material functions in the two domains are equivalent, but when the prescribed strain or stress history is a harmonic function of time, the constitutive relationships of the viscoelastic solid assume a very simple algebraic form with the constitutive parameters being the complex transversal and bulk moduli. From the knowledge of the complex moduli one can easily determine phase velocity and attenuation of harmonic waves propagating in linear viscoelastic media using the *elastic-viscoelastic correspondence principle* (Fung, 1965). Thus, wave propagation in linear viscoelastic unbounded media is completely described either by the complex-valued phase velocities V_P^* and V_S^*, or by the real-valued phase velocities, V_P and V_S, and attenuation factors α_P and α_S. It is easy to show that these parameters are related via the following expressions (Lai & Rix, 2002):

$$V_\chi^*(\omega)=\frac{V_\chi(\omega)}{2\cdot\sqrt{1+Q_\chi^{-2}(\omega)}}\cdot\left[1+\sqrt{1+Q_\chi^{-2}(\omega)}+i\cdot Q_\chi^{-1}(\omega)\right] \qquad Q_\chi(\omega)=\left[\frac{1-\left(\dfrac{\alpha_\chi(\omega)\cdot V_\chi(\omega)}{\omega}\right)^2}{\dfrac{2\cdot\alpha_\chi(\omega)\cdot V_\chi(\omega)}{\omega}}\right] \tag{1}$$

where $\chi = P, S, R$ denotes longitudinal, transversal and Rayleigh waves respectively, $i=\sqrt{-1}$ and ω is the circular frequency. Equation (1) is exact and therefore it is valid for any value of the quality factor $Q_\chi(\omega)$. Furthermore it shows that as a result of material dispersion $V_\chi(\omega)$ and $Q_\chi(\omega)$ are not independent and thus they may not be specified arbitrarily.

STRONGLY COUPLED RAYLEIGH INVERSION

The technique illustrated in Lai & Rix (2002) for the solution of the Rayleigh eigenproblem in a rather general, linear viscoelastic solid has been applied to develop a strongly coupled inversion algorithm of Rayleigh dispersion and attenuation curves. The inversion is performed by applying the complex formalism to a constrained least squares algorithm that enforces maximum *smoothness* to the resulting profile of complex-valued phase velocities $V_S^*(y)$. This type of *local-search* algorithm has been shown to be more robust for the solution of non-linear inverse problems than the traditional least squares-based algorithms (Constable et al., 1987). The method of *Lagrange multipliers* is employed to solve this complex-valued constrained minimization problem resulting in:

$$\boldsymbol{V}_S^* = \left\{ \mu\left(\partial^T \partial\right) + \left[\boldsymbol{W}^* \cdot \left(\boldsymbol{J}_S^*\right)_{V_{S0}^*}\right]^H \cdot \left[\boldsymbol{W}^* \cdot \left(\boldsymbol{J}_S^*\right)_{V_{S0}^*}\right]\right\}^{-1} \cdot \left[\boldsymbol{W}^* \cdot \left(\boldsymbol{J}_S^*\right)_{V_{S0}^*}\right]^H \cdot \boldsymbol{W}^* \bar{\boldsymbol{d}}_0^* \tag{2}$$

where μ is the Lagrange multiplier which may be interpreted as a smoothing parameter, $\boldsymbol{W}^*$ is a complex-valued, diagonal matrix formed by the reciprocal of the *standard deviations* associated with the data $\boldsymbol{V}_R^*$, ∂ is a matrix defining the *two-point-central* finite difference operator, $\bar{\boldsymbol{d}}_0^* = (\boldsymbol{J}_S^*)_{V_{S0}^*} \cdot \boldsymbol{V}_{S0}^* + \left(\boldsymbol{V}_R^* - \boldsymbol{V}_{R0}^*\right)$ and finally $\boldsymbol{J}_S^*$ is the complex-valued *Jacobian* matrix. One of the relevant features of this algorithm is the use of closed-form analytical expressions to compute the partial derivatives of the complex-valued Rayleigh phase velocity with respect to the complex-valued speed of propagation of body waves. These quantities were computed by combining the Rayleigh variational principle with an application of the integral transform methods, and their use contribute to enhance the efficiency of the inversion algorithm. The strongly coupled inversion algorithm is superior with respect to an uncoupled or weakly coupled analysis (Rix et al., 2000). First it takes into account in a elegant fashion the inherent coupling existing in a causal propagation of seismic waves between phase velocity and attenuation. Secondly, the strongly coupled inversion constitutes a better-posed mathematical problem (*in the sense of Hadamard*) as for the issue of solution uniqueness. This result can be attributed to a built-in constraint embedded in the formalism of the complex inversion and constituted by the Cauchy-Riemann equations satisfied by the Rayleigh phase velocity regarded as holomorphic function of $\boldsymbol{V}_S^*$. Figure 1 illustrates the results of the application of the inversion algorithm using a numerical simulation.

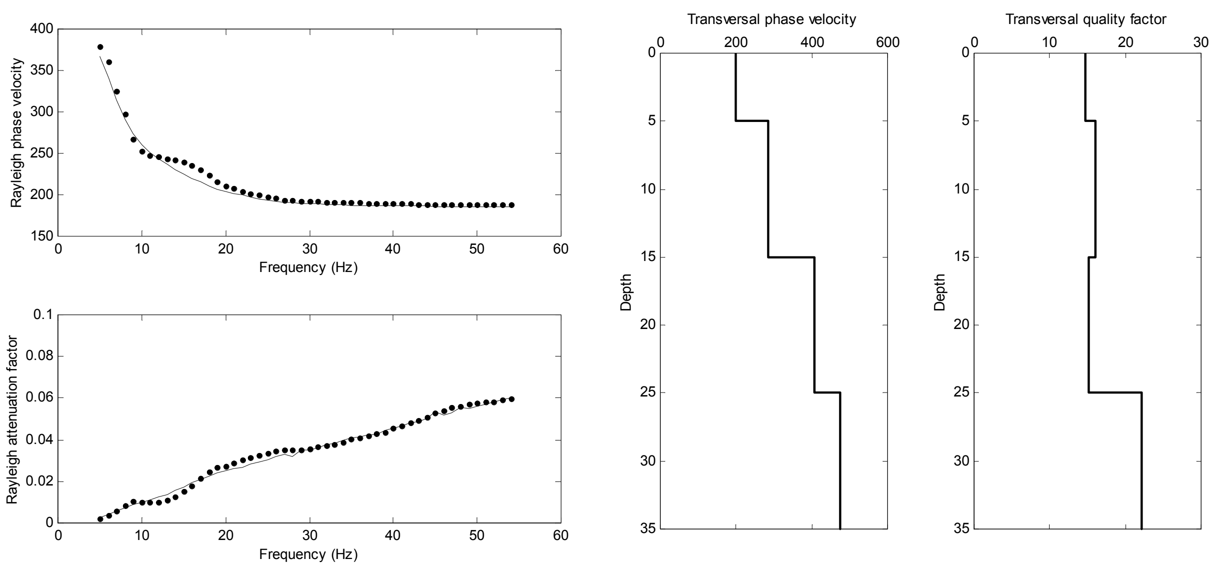

Figure 1 Strongly coupled inversion of Rayleigh dispersion and attenuation curves using a numerical simulation

CONCLUSIONS

This memoir has briefly illustrated an elegant procedure for the simultaneous inversion of surface wave data to obtain the phase velocity and quality factor profiles $V_S(y)$ and $Q_S(y)$ of transversal waves in a arbitrarily dissipative, linear viscoelastic continuum. An advantage of the algorithm is that it explicitly recognizes the inherent coupling existing in viscoelastic media between speed of propagation of body waves and attenuation as a result of material dispersion. Thus the algorithm is causal and it is more stable if compared with corresponding uncoupled analyses.

References

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