MATHEMATICAL AND NUMERICAL MODELING OF ELASTIC-PLASTIC WAVES IN GRANULAR MATERIALS

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Summary A process of shock waves propagation in elastic-plastic granular medium under small strains is described on the basis of new rheological model taking into account different resistance of material with respect to tension and compression. Some exact solutions of the problem of impact loading are obtained in one-dimensional case. By means of numerical experiments it is shown that plane fronts of two waves, bending due to inhomogeneous loosening, can be reflected with the formation of transverse cumulative splash.

INTRODUCTION

Mathematical theory of granular materials is an intensively developing field of mechanics. In spite of the fact that one of the first papers in this field [1] was published almost a hundred years ago, mathematical models of such materials are far from being completed. At present, there exist two classes of models related to two different regimes: the regime of quasistatic deformation and the regime of fast motion; see e.g. [2]. Models of the first class describe a behavior of compact particles in contact among themselves on the basis of plastic flow theory with the help of Coulomb–Moor or Mises–Schleicher criteria. Models of the second one deal with loose medium and simulate it as an ensemble of particles similar to kinetic gas theory.

However, the universally recognized model, similar to Navier–Stokes system in hydrodynamics of viscous flow or Hook law in the theory of elasticity, is absent nowadays. There isn't exist also any perceptible model of mixed type which can be used both in case of fast motion and that of quasistatic loading to describe the stagnant zones in a flow.

In the present paper we propose a geometrically linear model of the spatial deformation of an elastic-plastic granular material, applicable under tension and compression.

MATHEMATICAL MODEL

For a phenomenological description of materials, having different resistance to tension and compression, the rheological method is supplemented by a new element – rigid contact (Fig. 1*a*). For compressing stresses, this element is not deformed. For zero stress, the strain can be an arbitrary positive value. Tensile stresses are inadmissible. The diagram in Fig. 1*b* corresponds to the model of elastic-plastic granular medium. In compression, such medium is either in elastic state or in plastic state, but in tension the stresses are equal to zero. The model of heteromodular elastic-plastic material (Fig. 1*c*) can be considered as a regularization of the last one convenient for numerical realization.

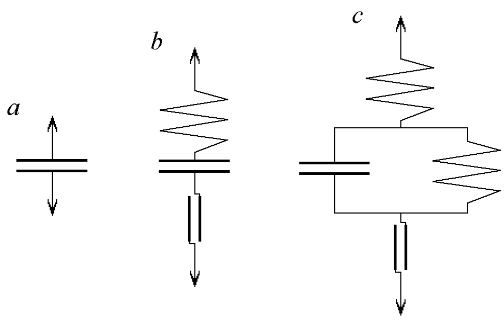

Figure 1. Rheological diagrams

Constitutive relationships of rigid contact, representing the diagram of ideal granular medium with rigid granules, reduce to pair of equivalent variational inequalities [3]

$$\sigma\,(\tilde\varepsilon-\varepsilon)\le 0,\quad \varepsilon,\,\tilde\varepsilon\in C;\qquad (\tilde\sigma-\sigma)\,\varepsilon\le 0,\quad \sigma,\,\tilde\sigma\in K,$$

where C and K are cones of nonnegative strains $\varepsilon \ge 0$ and nonpositive stresses $\sigma \le 0$, $\tilde\varepsilon$ and $\tilde\sigma$ are arbitrary varied quantities. These inequalities can be generalized to the case of a spatial stress-strained state by appropriate determination of cones in tensor spaces.

We consider a model of isotropic granular material whose elastic properties are characterized by bulk modulus k and shear modulus μ. For the description of admissible stresses Mises–Schleicher cone K is used. The yield surface is approximated by Mises cylinder F. It is shown that the stress tensor can be obtained as a projection onto cone K of conditional-stress tensor which is determined from Hook law over elastic part of strain tensor. Its plastic part can be found from associative flow rule. The closed mathematical model consists of the system of equations of motion, kinematic relationships and constitutive ones.

PLANE WAVES

A typical configuration of discontinuities arising in an elastic-plastic medium due to impact loading in the absence of initial stresses contains two shock waves: the elastic precursor, propagating with velocity of longitudinal elastic waves $c_p = \sqrt{(k+4\mu/3)/\rho}$ (ρ is mass density), behind the front of which stresses tend to the yield surface, and the plastic wave with velocity $c_f = \sqrt{k/\rho}$. This configuration is observed in a densified granular medium. In a loosened medium, the precursor wave gives way to an elastic signoton, i.e. plane longitudinal shock wave, on the front of which the strain changes its sign. Elastic signoton moves with the lower velocity $c_\tau = c_p\sqrt{-\varepsilon_\tau/(\varepsilon_0-\varepsilon)} < c_p$, where $\varepsilon_\tau = -\sqrt{3}\tau/(2\mu)$ is the limit strain of elastic compression, τ is the yield point. If the strain ε_0 ahead of the front exceeds limit value $\varepsilon_f = 2\tau/(\sqrt{3}k)$ then the velocity of signoton-precursor becomes less than c_f. The shock wave turns over, and the two-wave configuration is replaced by a solitary plastic signoton moving with the velocity $c = c_f\sqrt{(\varepsilon_f-\varepsilon)/(\varepsilon_0-\varepsilon)} < c_f$, depending on the reached strain ε in normal direction.

NUMERICAL ALGORITHM

Let U and V be vector-functions composed of mass velocity components in Cartesian coordinate system and components of the tensors of conditional stresses and real stresses, respectively. In such terms, the model is transforms to the variational inequality

$$(\widetilde{V}-V)\Big(AU_{,t}-\sum_{i=1}^{n}B^iV_{,i}-G\Big)\geq 0, \qquad V,\,\widetilde{V}\in F, \qquad V=\lambda U+(1-\lambda)U^\pi,$$

where A and B^i are symmetric matrices, G is the vector of mass forces, subscripts denote partial derivatives with respect to time and spatial variables, superscript π denotes the projection onto K, $\lambda\in(0,1)$ is the regularization parameter.

Numerical realization of such model is carried out on the basis of a combination of the space-variable decomposition method, on each stage of which one-dimensional problems are solved by means of monotone essentially nonoscillatory finite-difference scheme of the second degree of accuracy, and the special procedure of stresses correction taking into account irreversible strains.

RESULTS OF COMPUTATIONS

Some computations of one-dimensional problem about plane waves propagation were fulfilled for the testing of suggested algorithm. The comparison of numerical results with exact solutions, obtained in [4], has shown a good correspondence both on wave velocities and on wave amplitudes.

By means of two-dimensional model the problem of cumulative reflection of signotons was solved. In this problem the mass of granular medium, loosened in horizontal direction x_1 by linear law $\varepsilon_1 = \omega\, x_2$ (x_2 denotes a vertical coordinate), is loaded symmetrically by two lateral impulses. Propagating through inhomogeneous medium the plane fronts of signotons progressively bend, slowing down in loose domain with respect to dense one. Unloading waves follow signotons through compacted material, therefore their fronts remain practically plane up to the moment of collision with reflected waves. In the point of signotons contact nearby the lower bound – rigid wall – typical cumulative splash appears, which extends with time in the line of upper bound – free surface of mass. Configurations of plastic zone for different time moments are represented in Fig. 2.

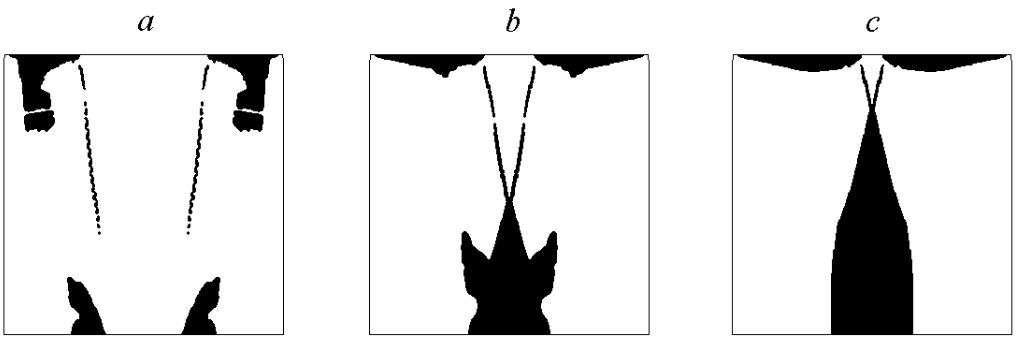

Figure 2. Cumulative interaction of signotons: (*a*) $t = 0.23$, (*b*) $t = 0.35$, (*c*) $t = 0.40$

ACKNOWLEDGEMENTS

This work was supported by grant of the President of Russian Federation no. MK-982.2004.1.

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