TOWARD CONVERGENCE IN INITIALLY RIGID COHESIVE FRACTURE MODELS

Chin-Hang Sam*, Katerina Papoulia*, Pritam Ganguly**
*School of Civil and Environmental Engineering, Cornell University, Ithaca, NY 14853, USA
**Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, USA

Summary We analyze the convergence of finite element methods for cohesive fracture. The focus is on generalized fracture in which finding the crack path or paths is part of the solution process. Initially rigid cohesive models, in which interface elements are inactive until a critical traction is attained, are usually preferred in this context. We show that convergence as the time-step tends to zero is erratic or nonexistent unless the model satisfies a property called "time continuity." We also argue that convergence as the spatial mesh size tends to zero is unlikely unless the mesh is able to represent all possible crack paths without preferred directions. We propose a method to achieve this kind of mesh isotropy in two dimensions based on Radin's pinwheel tiling. The pinwheel tiling in the limit can approximate any plane curve with the correct length.

COHESIVE FINITE ELEMENTS

Cohesive zone modeling, which was originally proposed by Dugdale and Barenblatt, represents fractures with explicit displacement jumps. In such a model, the separation of bulk material is resisted by cohesive forces governed by the corresponding cohesive constitutive model.

It is natural to apply cohesive modeling to problems where defined interfaces exist and, indeed, this approach has been successfully used in problems concerning bi-material interfacial debonding and fracture, delamination of composite materials, and many other settings. A widely used cohesive model is one that consists of an ascending branch and a descending branch, referred to here as "initially elastic."

In the past decade, there are increasing applications of cohesive finite elements in the dynamic fracture arena, including void nucleation, interfacial fracture, branching and fragmentation. In some of these simulations, the crack pattern is not known in advance. This precludes the possibility of prepositioning cohesive surfaces. Alternatively, every edge of the bulk elements is considered as a potential fracture surface. The crack propagation path can then be resolved as part of the solution of the governing equations.

Two different approaches are taken to this end. In the first approach, every edge of each bulk element is the site of an initially elastic cohesive surface. A fundamental problem is that as the spatial discretization is refined, the effective modulus of the material is non-physically decreased ultimately to zero [1].

Trying to solve this problem leads to the second approach, namely an adaptive approach in which cohesive surfaces are only inserted when they are needed. In other words, the effective initial stiffness of the cohesive model is infinite. We denote this type of model as initially rigid. Although the initially rigid model can eliminate some drawbacks of the initially elastic model, special considerations in its finite element implementation are essential to its numerical behavior, and it was pointed out by Papoulia et al. [3] that in explicit dynamics using a rigid model, the correct time convergence rate can be obtained only if a certain condition called "time continuity" is satisfied.

In this paper, we propose an improved model that also satisfies the time-continuity condition. The rigid cohesive modeling methodology and the time-continuity condition are briefly described in Section 2.

Section 3 considers convergence of the initially rigid cohesive model as the spatial mesh size is refined. It is argued that many typical meshes are poorly suited for cohesive zone modeling for generalized fracture because these meshes contain unphysical preferred crack directions. The existence of these preferred directions precludes convergence to the true solution as the mesh size tends to zero. A two-dimensional technique based on Radin's pinwheel tiling is proposed for remedying this problem. Pinwheel tilings have the property that given any piecewise smooth plane curve P and any $\epsilon > 0$, there is a sufficiently fine refinement R of the mesh such that R contains a path P' of mesh edges such that P' lies within distance ϵ of P and such that $|\text{length}(P') - \text{length}(P)| \leq \epsilon$.

TIME CONVERGENCE

In this section, the matter of time continuity and convergence as the time-step tends to zero is analyzed. For this section, regard cohesive fracture as a spatially discretized differential equation in the standard form of nonlinear frictionless mechanics, namely,

$$\mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}^{ext} - \mathbf{f}^{int}(\mathbf{u}, \mathbf{q}) \equiv \mathbf{f}, \tag{1}$$

where $\mathbf{M}$ is a mass matrix, $\mathbf{u}$ is the global nodal displacement vector, $\mathbf{f}^{int}$ is the internal nodal force vector and $\mathbf{f}^{ext}$ the external nodal force vector. $\mathbf{f}^{int}$ can explicitly depend on the displacement vector $\mathbf{u}$ and some history vector $\mathbf{q}$. In order to obtain convergent results as the time step is varied, $\mathbf{f}^{int}(\mathbf{u}, \mathbf{q})$ is required to be a continuous function of $\mathbf{u}$ for those values of $(\mathbf{u}, \mathbf{q})$ encountered on the solution trajectory of Equation (1). Specifically, quadratic convergence is obtained with the central difference time stepping scheme when time-continuity is observed, whereas lack of convergence or erratic sequences of interface activations are observed when the condition is violated [3]. The source of discontinuities of the internal force vector arise at the time step when a new cohesive surface is inserted. Since the bulk constitutive model and the cohesive constitutive model are chosen independently, continuity of the nodal forces before and after the insertion is not guaranteed. We propose a methodology that converts almost any initially elastic cohesive into a time continuous initially rigid model. The methodology is simpler and more general than the approach to time continuity taken by [3].

The new approach is tested in a mixed mode dynamic fracture experiment of a concrete beam investigated by John and Shah [2] in which a notched specimen supported at two corners is subjected to an impact load. Different time step values Δt_i are used ($\Delta t_i = 2 \times 10^{-7}/2^i$ s, $i = 0 : 5$) and the results from the simulation using the smallest time step (i.e. $i = 5$) is taken as the "exact" solution. The relative error is defined as $\text{error}_i = \| \mathbf{v}_i - \mathbf{v}_{exact} \|_2 / \| \mathbf{v}_{exact} \|_2$, where $\mathbf{v_i}$ is the velocity vector of the simulation using Δt_i. The convergence results of the time-continuous model TCM (marked by 'o') and discontinuous model TDM (marked by 'x') are compared at two different times ($t_1 = 2 \times 10^{-4}$ s as the dashed line, $t_2 = 6.8 \times 10^{-4}$ s as the dotted line) and is shown in Figure 1 with logarithmic scale. For a convergent method, one expects that the error will diminish to zero as the time step size goes to zero, and quadratic convergence means that the rate of decrease will be of second order. Hence, it can be observed that the results of the continuous model shows quadratic convergence for time steps smaller than a certain critical value, but for the discontinuous model, the results are either converging at a very slow rate or not converging at all. In addition, while the activated interfaces in the continuous model have similar patterns for all the different choices of time step, the activated interfaces in the discontinuous model are different for each time step size.

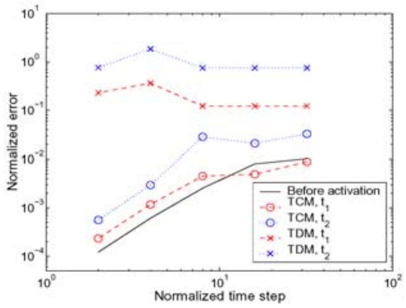

Figure 1. Convergence results of mixed mode beam bending.

SPATIAL CONVERGENCE

Although several researchers have carried out spatial convergence studies with cohesive models, there is no proof available that cohesive finite element models converge to a "true" solution as the spatial mesh size tends to zero. Indeed, we claim that such convergence is not likely unless a special kind of mesh is used. The reason is that standard meshes used in the literature contain preferred fracture paths. Consider, e.g., a cross-triangle quadrilateral mesh in which the quadrilaterals are squares. In both the true solution and the finite element solution, the energy required to form a crack is proportional to its length. If the true crack path is linear and is oriented at 0°, 45°, 90° or 135° with respect to the x-axis, then this crack path is exactly represented in the mesh as the mesh is refined. On the other hand, if the crack has any other orientation, e.g., 22.5°, then its length is not exactly represented as the mesh is refined. In fact, there is a constant $\alpha > 1$ such that the ratio between the length of the mesh's representation of the crack and true crack length is greater than α for all levels of refinement of this mesh. Therefore, the energy to form this crack never converges to the true energy as the spatial mesh is refined. This persistent gap in the energy means that cohesive finite elements will probably never find the true crack solution in this mesh if it is a line oriented at 22.5°.

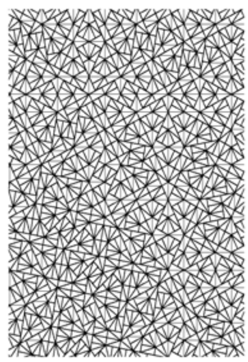

Figure 2. Radin's pinwheel tiling of the plane

We propose a technique to solve this problem based on Radin's pinwheel tiling of the plane [4], a portion of which is depicted in Fig. 2. Radin's tiling has the property that any piecewise smooth curve is represented arbitrarily accurately (including its length) by tile edges as the tiling is refined. Radin's pinwheel tiling is not suitable for use in finite element modeling because it will not conform to boundaries. We present computations using a generalization of the pinwheel tiling for arbitrary 2D domains that preserves the length-approximation property.

References

[1] M. L. Falk, A. Needleman, and J. R. Rice. A critical evaluation of cohesive zone models of dynamic fracture. Journal de Physique IV, Prceedings, Pr-5-43-Pr-5-50, 2001.

[2] R. John and S. P. Shah. Mixed-mode fracture of concrete subjected to impact loading. Journal of Structural Engineering, 116(3):585–602, 1990.

[3] K. Papoulia, C. Sam, and S. Vavasis. Time continuity in cohesive finite element modeling. International Journal for Numerical Methods in Engineering, 58(5):679–701, 2003.

[4] C. Radin. The pinwheel tilings of the plane. Annals of Math., 139:661–702, 1994.