Flutter of Cantilever Wings

A classical example of non-self-adjoint problems consists of the combined bending and torsional vibration of a cantilever aircraft wing in steady air flow shown in Fig. 1.

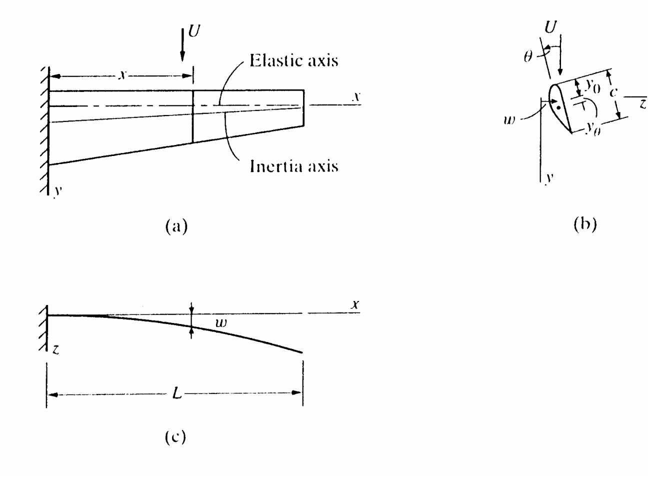

Figure 1. (a) Elastic axis and inertia axis for a cantilever aircraft wing in steady air flow; (b) Wing cross section; (c) Bending deflection of the elastic axis.

The boundary-value problem for the free vibration of the wing in the presence of aerodynamic forces is described by the following differential equations:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2W}{dx^2}\right)+\frac{rU^2}{2}c\frac{dC_L}{dq}\Theta+l\frac{rU}{2}c\frac{dC_L}{dq}\left[W+c\left(\frac{3}{4}-\frac{y_0}{c}\right)\right]\Theta+l^2m(W+y_p\Theta)=0 \quad 0<x<L$$

$$\frac{d}{dx}\left(GJ\frac{d\ \Theta}{dx}\right)+\frac{rU^2}{2}c^2\left(\frac{1}{4}-\frac{y_0}{c}\right)\frac{dC_L}{dq}\Theta+l\frac{rU}{2}c^2\left\{\left(\frac{1}{4}-\frac{y_0}{c}\right)\frac{dC_L}{dq}W+c\right. \qquad 0<x<L$$
$$\left.\left[\left(\frac{1}{4}-\frac{y0}{c}\right)\left(\frac{3}{4}-\frac{y0}{c}\right)\frac{dC_L}{dq}+\frac{\Pi}{8}\right]\Theta\ \right\}+l^2(my_q(W+I_q)=0$$

The differential eigenvalue problem (with consideration of the boundary conditions), has no closed-form solution, so that we consider an approximate solution by means of Galerkin's method. To this end, we assume a solution in the form

$$W(x)=\mathrm{f}_1^T(x)\mathbf{a}_1,\ \Theta(x)=\mathrm{f}_2^T(x)\mathbf{a}_2$$

With introduction of the boundary conditions and after integrating over the length of the beam, we obtain the following algebraic eigenvalue problem:

$$\left\lfloor K+U^2H+lUL+l^2M\right\rfloor \mathbf{a}=\mathbf{0}$$

where $\mathbf{a}=\left[\ \mathbf{a}_1^T \qquad \mathbf{a}_2^T\ \right]^T$ and the various matrices have the submatrices

$$K_{11}=\int_0^L f_1(EIf_1''^T)''dx=\int_0^L EIf_1''f_1''^T dx$$
$$K_{12}=0 \qquad\qquad K_{21}=0$$
$$K_{22}=-\int_0^L \mathrm{f}_2(GJ\mathrm{f}_2'^T)'dx=\int GJ\mathrm{f}_2'\mathrm{f}_2'^T dx \quad H_{11}=0 \quad \ldots\ldots..$$

The eigenvalue l is a continuous function of the air speed U. When $U=0$, the system is conservative and l is pure imaginary. For $U\neq 0$, $l=a+iw$. It can be shown that for sufficiently small U and for $\frac{dC_L}{dq}<2p$ the wing is losing energy to the surrounding air, so that the motion represents damped oscillation. This implies asymptotic stability, so that $a<0$. At some point, as U increases, a turns from negative to positive, as shown in Fig. 2, so that the motion turns from asymptotically stable to unstable. At point $a=0$, at which the motion is merely stable and ready to become unstable, the air speed reaches the critical value U_{cr}.

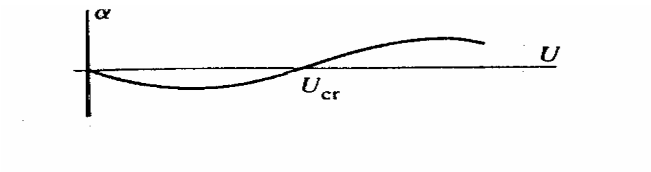

Fig.2 The real part of eigenvalue vs. the air speed.

After some developpement, we obtain the quadratic equation in U_{cr}^2

$$aU_{cr}^4 + bU_{cr}^2 + c = 0$$

With the simplified solution:

$$U_{cr}^2 = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$$

Validation:

In order to validate the accuracy of the model developed so far, Goland's wing was used. The flutter result in incompressible flow by using the quasi-steady method was calculated, which is compared with the exact one (Jensen Lin , NASA/TM 2000).
Consider a wing with constant chord and the following characteristics (Fig.1):

$$L = 10 \quad m \qquad c = 3 \quad m \quad y_{cg} = 0.6c \quad y_0 = 0.5c \quad m = 200 \; kg \qquad \frac{dC_L}{dq} = 2p$$

$r = 1.220 \quad Kg / m^3$ Applying the approach developed in this paper it was found that: $w_F = 96 \; rad / s$ and $V_F = 191 \; m / s$ as shown in Fig. 3

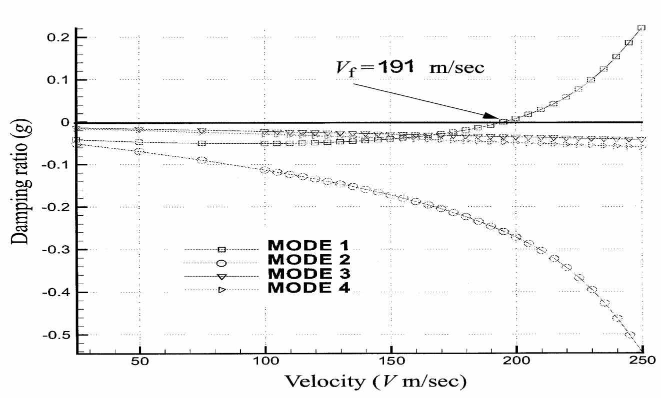

Fig.3 Velocity vs. damping

Table 1 Comparison of the calculated flutter results

Method	Description	Flutter Speed (Mach#)	Flutter frequency (HZ)
Exact	2-D incompres. flow	0.554	11.25
Galerkin	2-D incompres. flow	0.554	11.15
Galerkin	2-D compres. flow	0.526	10.90