Agradient velocity and vortical motion in rotating fluids

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A new approach to modelling slow vortical motion and fast inertia-gravity waves is suggested within the rotating shallow-water equations (SWE) with arbitrary topography.

Agradient velocity. In the momentum-balance equation the agradient velocity, $\mathbf{v}$, proportional to the velocity tendency is introduced as the difference between the velocity, $\mathbf{V}$, and the gradient wind, expressed via absolute vorticity, A, and Bernoulli function, B

$$\mathbf{v} \equiv A^{-1}\mathbf{k}\times\partial_t\mathbf{V} = \mathbf{V} - A^{-1}\mathbf{k}\times\nabla B. \tag{1}$$

According to definitions of A and B, we have

$$A - f = \nabla\cdot(A^{-1}\nabla B) + \mathbf{k}\cdot(\nabla\times\mathbf{v}), \tag{2}$$

$$B = P - \overline{P} + E, \qquad E = \frac{1}{2}(A^{-1}\nabla B - \mathbf{k}\times\mathbf{v})^2. \tag{3}$$

Here t is the time, $\mathbf{V} = (U, V)$ is the horizontal velocity vector in the local tangential plane at the rotating planet, $\mathbf{k}$ is the vertical unit vector, $f = 2\Omega_E \sin\theta$ is the Coriolis parameter, Ω_E is the rotational frequency of the planet, θ is the latitude, $P = gH$ is the geopotential height, g the gravity acceleration, $\overline{P} = g\overline{H}$, and $\overline{H}$ is the unperturbed fluid depth which may include bottom topography. Recall that substitution of a reduced gravity g' in the place of g, makes the SWE applicable to one active layer of a two-layer fluid with no free surface and an infinitely deep second layer at rest; such an equivalent barotropic system is often used for oceanic and atmospheric applications. With the operator notations the SWE has the same form either on a hemisphere or in local cartesian coordinates (x, y) where $\nabla = (\partial/\partial x, \partial/\partial y)$.

The material invariant in the case of conservative dynamics, the shallow-water potential vorticity (PV), A/P, as well as the inverse value, $Q = P/A$, satisfies to the equation

$$\frac{DQ}{Dt} \equiv (\partial_t + \mathbf{V}\cdot\nabla)Q = \partial_t Q + \mathbf{v}\cdot\nabla Q + A^{-1}J(B, Q) = 0. \tag{4}$$

Here the Jacobian operator is $J(B, Q) \equiv \mathbf{k}\cdot(\nabla B \times \nabla Q)$. In this form we obtain that the gradient wind advection of Q is described by $J(B, Q)$ similar to the quasigeostrophic model.

The agradient velocity evolution is described by the equation obtained by differentiating Eq. (1) by time and excluding $\partial_t A$, $\partial_t B$, that gives

$$A\mathbf{k}\times\partial_t\mathbf{v} - L(\mathbf{v}) = -\nabla J(B, Q), \qquad L(\mathbf{v}) \equiv A^2\mathbf{v} + \nabla\cdot(A\mathbf{v})\frac{\nabla B}{A} - \nabla[\mathbf{v}\cdot\nabla B + \nabla\cdot(AQ\mathbf{v})]. \tag{5}$$

The second-order system of equations (5) for two components of agradient velocity serves as an alternative form of the exact equations for the first time derivatives of divergence and ageostrophic vorticity. It also describes inertia-gravity waves propagating through inhomogeneous media if the background variables denoted by capital letters are given. The source term here is expressed by $\nabla J(B, Q)$ which is related to the gradient of Q evolving according to the first-order equation (4). In this form, the intrinsic source of adiabatic flow evolution is expressed as a single term $J(B, Q)$ (for any steady flow $J(B, Q) = 0$).

Vortical motion This approach allows for the construction of a hierarchy of balance relations for vortical dynamics and potential vorticity inversion schemes even for moderate Rossby and Froude numbers assuming the characteristic value of $|J(B,Q)| = \epsilon$ to be small. From (5) we see that agradient velocity is scaled by ϵ being proportional to $\nabla J(B,Q)$. Therefore, in the lowest-order approximation we may neglect $\mathbf{v}$ in (2)–(4) that gives the potential vorticity-conserving gradient wind balance model where absolute vorticity $\hat{A}$ and Bernoulli function $\hat{B}$ are related through the elliptic equation (2) if the condition $\hat{A} > 0$ is satisfied.

If we compare the solution of the gradient wind balance model to the full SWE solution described by (1)–(5) started with the same initial gradient wind (initially zero agradient velocity), we will see the development of agradient velocity with unbalanced inertia-gravity waves generated by $\nabla J(\hat{B},Q)$ in (5). The important advantage of (5) is that the wave amplitude can be estimated from (5) as $\epsilon \ll 1$. Thus, fast propagation of inertia-gravity waves can be considered in (5) at the slowly varying inhomogeneous background described by balanced fields $\hat{A}$, $\hat{B}$, Q in the same fashion as in the geostrophic adjustment problem. The slow component of the solution to (5) is described by the balance condition for agradient velocity obtained by setting $\partial_t \mathbf{v} = 0$

$$L(\mathbf{v}) \equiv \hat{A}^2\hat{\mathbf{v}} + \nabla\cdot(\hat{A}\hat{\mathbf{v}})\frac{\nabla\hat{B}}{\hat{A}} - \nabla[\hat{\mathbf{v}}\cdot\nabla\hat{B} + \nabla\cdot(\hat{A}Q\hat{\mathbf{v}})] = \nabla[J(\hat{B},Q)]. \tag{6}$$

In this next order, the components of agradient velocity are used as the fast variables slaved to potential vorticity that allows for diagnostic estimates of the velocity tendency, the direct potential vorticity inversion with the accuracy of ϵ^2 and the corresponding potential vorticity-conserving agradient velocity balance model (AVBM).

The coupled system (6) for components of balanced agradient velocity, $\hat{\mathbf{v}}$, is quasilinear, and it can be transformed into a single equation for the balanced tendency of Bernoulli function $\partial_t\hat{B}$ from

$$\hat{P}\hat{\mathbf{v}} - [\hat{\mathbf{v}}\cdot(\nabla Q + \frac{\nabla\hat{B}}{\hat{A}})]\frac{\nabla\hat{B}}{\hat{A}} = [\partial_t\hat{B} + J(\hat{B},Q)]\frac{\nabla\hat{B}}{\hat{A}^2} - \frac{Q}{\hat{A}}\nabla\partial_t\hat{B}. \tag{7}$$

Here $\hat{P} = \hat{A}Q = C_g^2$ and the determinant in the left hand side is defined by

$$D = C_g^2[C_g^2 - \hat{A}^{-2}(\nabla\hat{B})^2 - \hat{A}^{-1}\nabla\hat{B}\cdot\nabla Q]. \tag{8}$$

Although, no apriori restrictions on the Rossby and Froude number were assumed, this form of balance relations reveals an ultimate limitation: the determinant D should be positive. It can be rewritten in the form

$$C_g^2 - \hat{\mathbf{V}}^2 + \hat{\mathbf{V}}\cdot\mathbf{V}_R > 0, \qquad \mathbf{V}_R \equiv -\mathbf{k}\times\nabla Q, \tag{9}$$

where C_g and $\mathbf{V}_R$ represent the local gravity wave and characteristic vortical Rossby wave phase velocity, correspondingly. Without the last term, (9) would describe subcriticality of the gradient wind that corresponds to Ripa's first formal criteria of the stability for localized vortical flows. The negativeness of the last term corresponds to his second formal criteria of stability. Therefore, (9) is satisfied if both formal criteria of stability hold.

The accuracy of the AVBM is illustrated by considering the linear normal modes, coastal Kelvin waves, nonlinear evolution of elliptic and dipolar vortices in the f-plane channel with topography.