TRANSIENT ANALYSIS OF PIEZOELECTRIC PLATES WITH THERMAL EFFECTS

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Summary In this paper a finite element method for transient analysis of piezoelectric plates with thermal effects is presented. The kinematics of the model is based on a higher order plate theory. In particular, the plate is assumed to be able of thickness distensions and the electric potential and the temperature can vary through the thickness. The variational statement which supports the finite element formulation is of mixed type, involving mechanical, electrical and thermal both primary and secondary variables. The time integration is performed based on a discontinuous Galerkin approach.

Due to their properties of transforming mechanical energy into electrical energy and vice versa [1], piezoelectric materials are widely used in building devices for smart structural control. Mostly, these devices are plates, often multilayered, and can be structural elements or specific elements connected to the structure. Owing to the pyroelectric effect, the response of these devices may change considerably when they are subjected to extensive thermal loads, like, for example, in the aerospace structures. Therefore, in order to correctly predict the devices response, it is necessary to accurately model not only the mechanical and electrical variables but also the thermal ones and the associated coupling effects. This provided, a finite element method for transient analysis of piezoelectric plates with thermal effects is presented here.
The basic field variables which describe the plate behavior are the displacement components u, v, w, the electric potential ϕ and the temperature rise θ from a reference value T_0. As regards the displacement, in order to incorporate the transverse shear effects, the inplane components are assumed according to the Reddy's third-order theory as

$$u(\mathbf{x},\varsigma,t) = u_0(\mathbf{x},t) - \varsigma\,\psi_1(\mathbf{x},t) - \varsigma^3 \frac{4}{3h^2}\left[\psi_1(\mathbf{x},t) + \frac{\partial w_0(\mathbf{x},t)}{\partial x_1}\right], \tag{1}$$

$$v(\mathbf{x},\varsigma,t) = v_0(\mathbf{x},t) - \varsigma\,\psi_2(\mathbf{x},t) - \varsigma^3 \frac{4}{3h^2}\left[\psi_2(\mathbf{x},t) + \frac{\partial w_0(\mathbf{x},t)}{\partial x_2}\right], \tag{2}$$

where a Cartesian reference frame $(O,\mathbf{x},\varsigma)$ is chosen with the origin O on the middle cross-section of the plate and the ς-axis in the thickness direction, (u_0, v_0, w_0) and (ψ_1, ψ_2) denote, respectively, the displacements and rotations of a transverse normal to the plane $\varsigma = 0$, h is the thickness of the plate and t the time. Moreover, following the idea proposed in [2][3], the plate is assumed to be capable of thickness oscillations, whose influence on the global plate behavior may result significant, for example, in some high-frequency resonators. Therefore, the transverse displacement component has to depend on the thickness coordinate ς, and, in particular, it is assumed as a cubic function of ς

$$w(\mathbf{x},\varsigma,t) = w_0(\mathbf{x},t) + \varsigma\, w_1(\mathbf{x},t) + \varsigma^2\, w_2(\mathbf{x},t) + \varsigma^3\, w_3(\mathbf{x},t). \tag{3}$$

Finally, the electric potential and the temperature are assumed, respectively, as a cubic and as a quadratic function of ς in the forms

$$\phi(\mathbf{x},\varsigma,t) = \phi_0(\mathbf{x},t) + \varsigma\,\phi_1(\mathbf{x},t) + \varsigma^2\,\phi_2(\mathbf{x},t) + \varsigma^3\,\phi_3(\mathbf{x},t), \tag{4}$$

$$\theta(\mathbf{x},\varsigma,t) = \theta_0(\mathbf{x},t) + \varsigma\,\theta_1(\mathbf{x},t) + \varsigma^2\,\theta_2(\mathbf{x},t). \tag{5}$$

Approximations (4) and (5) permit to accurately satisfy the electrical and the thermal boundary conditions on the plate surfaces. In addition, they, together with the approximation chosen for w, are in accordance with what the consistency analysis [4][5] suggests in order to achieve a fully thermopiezoelectric coupling.
The constitutive equations used to develop the plate model are those of the fully coupled thermopiezoelectricity, which can be written as

$$\mathbf{D}_u\mathbf{u} = \mathbf{H}_u\boldsymbol{\sigma} + \mathbf{h}_{u\phi}^{\mathrm{T}}\mathbf{d} + \mathbf{h}_{u\theta}\boldsymbol{\theta}, \quad \mathbf{D}_\phi\boldsymbol{\phi} = \mathbf{H}_\phi\mathbf{d} - \mathbf{h}_{u\phi}\boldsymbol{\sigma} - \mathbf{h}_{\theta\phi}\boldsymbol{\theta}, \quad \mathbf{D}_\theta\boldsymbol{\theta} = -\mathbf{H}_\theta\mathbf{q} \tag{6}$$

together with the constitutive assumption for the entropy

$$s = \mathbf{h}_\theta^{\mathrm{T}}\boldsymbol{\theta} + \mathbf{h}_{\theta\phi}^{\mathrm{T}}\mathbf{d} + \mathbf{h}_{u\theta}^{\mathrm{T}}\boldsymbol{\sigma} \tag{7}$$

which is used to express the dissipative term in the thermal balance equation. In the above equations,

$$\mathbf{u} = [u_0, v_0, w_0, \psi_1, \psi_2, w_1, w_2, w_3]^{\mathrm{T}}, \quad \boldsymbol{\phi} = [\phi_0, \phi_1, \phi_2, \phi_3]^{\mathrm{T}}, \quad \boldsymbol{\theta} = [\theta_0, \theta_1, \theta_2]^{\mathrm{T}},$$

$\mathbf{D}_u$, $\mathbf{D}_\phi$ and $\mathbf{D}_\theta$ are the differential operators of mechanical, electrical and thermal compatibility, respectively, $\boldsymbol{\sigma}$, $\mathbf{d}$ and $\mathbf{q}$ are vectors that collect the generalized stresses, electric flux density components and heat flux components, respectively, $\mathbf{H}_u, \mathbf{H}_\phi, \mathbf{H}_\theta, \mathbf{h}_\theta, \mathbf{h}_{u\phi}, \mathbf{h}_{u\theta}$ and $\mathbf{h}_{\theta\phi}$ are the thermopiezoelectric constitutive matrices. Equations $(6)_{1,2}$ show that there is a fully coupling between the mechanical and electrical variables, and equation (7) couples these with the thermal ones.

Within this framework, a mixed finite element formulation is developed. Mixed-type approaches have been successfully employed in treating thermoelasticity [6][7] as well as piezoelectricity [8]-[10] problems. The resultant procedures of analysis have shown to be accurate and convenient for the possibility of controlling separately all the quantities involved in a process. The variational statement which supports the formulation can be written as

$$\int_I \delta[\Pi^E + \Pi^T + \Pi^K + \Pi_{\text{ext}}]\,\mathrm{d}t - \int_I D\,\mathrm{d}t + \int_I G\,\mathrm{d}t - C = 0 \qquad \forall(\delta\boldsymbol{\sigma}, \delta\mathbf{u}, \delta\mathbf{q}, \delta\boldsymbol{\theta}, \delta\mathbf{d}, \delta\boldsymbol{\phi}), \tag{8}$$

where Π^E and Π^T are Hellinger-Reissner-type functionals for piezoelectricity and heat conduction problems,

$$\Pi^E(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{d}, \boldsymbol{\phi}) = \int_\Omega (-\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{H}_u\boldsymbol{\sigma} + \frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{H}_\phi\mathbf{d} - \mathbf{d}^{\mathrm{T}}\mathbf{h}_{u\phi}\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{D}_u\mathbf{u} - \mathbf{d}^{\mathrm{T}}\mathbf{D}_\phi\boldsymbol{\phi})\,\mathrm{d}V, \tag{9}$$

$$\Pi^T(\mathbf{q}, \boldsymbol{\theta}) = \int_\Omega (-\frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}_\theta\mathbf{q} - \mathbf{q}^{\mathrm{T}}\mathbf{D}_\theta\boldsymbol{\theta})\,\mathrm{d}V, \tag{10}$$

Π^K is the kinetic energy,

$$\Pi^K(\mathbf{u}) = \frac{1}{2}\int_\Omega \dot{\mathbf{u}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{u}}\,\mathrm{d}V, \tag{11}$$

functional D takes into account the thermal distortions and the effects of the pyroelectric coupling,

$$D(\boldsymbol{\theta}, \delta\boldsymbol{\sigma}, \delta\mathbf{d}) = \int_\Omega (\delta\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{h}_{u\theta}\boldsymbol{\theta} + \delta\mathbf{d}^{\mathrm{T}}\mathbf{h}_{\theta\phi}\boldsymbol{\theta})\,\mathrm{d}V, \tag{12}$$

functional G introduces the thermoelastic and thermoelectric couplings and the rate of thermal energy,

$$G(\boldsymbol{\sigma}, \mathbf{d}, \boldsymbol{\theta}, \delta\boldsymbol{\theta}) = \int_\Omega \delta\boldsymbol{\theta}^{\mathrm{T}}\left(T_0\mathbf{h}_\theta^{\mathrm{T}}\dot{\boldsymbol{\theta}} + T_0\mathbf{h}_{\theta\phi}^{\mathrm{T}}\dot{\mathbf{d}} + T_0\mathbf{h}_{u\theta}^{\mathrm{T}}\dot{\boldsymbol{\sigma}}\right)\,\mathrm{d}V, \tag{13}$$

the term C serves to relax the initial conditions, Π_{ext} is the potential energy of the external mechanical, electrical and thermal loads, $\mathbf{M}$ is the mass matrix, I is the time interval and a superposed dot denotes differentiation with respect to time.

In the finite element discretization, the dynamic framework is phrased by reviewing the consistent approaches successfully employed in [7] for the thermoelastic case and in [9][10] for the piezoelectric one. Displacement, electric potential and temperature within the element and those at the interelement are represented independently of each other. In this way, the proposed formulation leads to element equations which can be expressed in terms of nodal displacements, electric potentials and temperatures only. Thus, conventional finite element procedures can be applied and the resulting finite elements can be readily implemented in existing codes. Finally, the integration in the time domain is performed based on the discontinuous Galerkin approach recently proposed in [11].

Preliminary numerical results obtained in some test cases are very encouraging.

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