NONLINEAR RESPONSE OF MAGNETOELASTIC SOLIDS

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Summary Recently, growing interest in magnetoelastic solids has motivated a renewed interest in electromagnetic continua with particular reference to large magnetoelastic deformations. We derive governing equations of equilibrium and constitutive laws expressed in either Lagrangian and Eulerian form. The equations are then applied to the solution of a prototype boundary-value problem.

MOTIVATION AND BASIC EQUATIONS

Recently, growing interest in the mechanical and the electromagnetic properties of composites consisting of an elastomeric matrix and a distribution of ferrous micron-sized particles embedded within their bulk has been observed. This interest is motivated by newly developed engineering applications, which involve, for example, sensors, vibration absorbers, and controllable membranes for use in civil and automotive engineering. See, for example, Ginder et al. [1] and Carlson and Jolly [2]. These materials have mechanical properties that can be altered rapidly by a change in the magnitude or direction of an applied magnetic field. The magnetic response is optimized by distributing within the bulk matrix particles with a high magnetic saturation, such as an alloy of iron, and volume fractions between 0.1 and 0.5. The choice of the matrix material is based on its thermomechanical properties and, for example, silicone and other elastomers are found to be suitable materials.

In this talk we first summarize the basic equations governing the magnetic field and its interaction with a deforming continuum. To derive the constitutive properties of the magnetoelastic composite we assume the existence of a free energy function, which depends on a deformation or strain measure in addition to a magnetic field variable. Following the description of nonlinear magnetoelastic deformations of elastomers given in recent papers by the present authors [3, 4, 5], we select the magnetic induction vector $\mathbf{B}$ and the deformation gradient tensor $\mathbf{F}$ as the basic variables and write the free energy $\Psi = \Psi(\mathbf{F}, \mathbf{B})$. Applying the Clausius-Duhem inequality, expressions for the Cauchy stress tensor $\boldsymbol{\sigma}$ (in general non-symmetric) and for the magnetization vector field $\mathbf{M}$ are derived. Secondly, we provide a Lagrangian counterpart of the free energy formulation such that $\Psi(\mathbf{F}, \mathbf{B}) = \Phi(\mathbf{F}, \mathbf{B}_0)$, where $\mathbf{B}_0$ is the Lagrangian vector field corresponding to $\mathbf{B}$. The free energy Φ is used to derive expressions for the symmetric total stress tensor $\boldsymbol{\tau}$ for both compressible and incompressible materials, with the appropriate specialization for isotropic material response. The total stress tensor $\boldsymbol{\tau}$ includes, in addition to the Cauchy stress $\boldsymbol{\sigma}$, the so-called Maxwell stress. The specialization of these equations to the situation where there is no deformation but only an applied magnetic field is considered and it is pointed out that the magnetic induction $\mathbf{B}_0$ generates a residual stress in the material.

Then, we introduce an augmented free energy formulation, denoted $\Omega = \Omega(\mathbf{F}, \mathbf{B}_0)$ and defined, for incompressible materials, by

$$\Omega = \rho_0 \Phi + \frac{1}{2}\mu_0^{-1}\mathbf{B}_0 \cdot (\mathbf{c}\mathbf{B}_0),$$

where ρ_0 is the mass density in the reference configuration, the constant μ_0 is the magnetic permeability in free space and $\mathbf{c} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green deformation tensor. This allows the magnetoelastic equilibrium equations to be written in a very compact form.
For incompressible materials, the total stress $\boldsymbol{\tau}$ can be written in the simple form

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I},$$

where p is a Lagrange multiplier associated with the constraint $\det \mathbf{F} \equiv 1$. The other relevant Eulerian quantities are given by

$$\mathbf{H} = \mathbf{F}^{-T}\frac{\partial \Omega}{\partial \mathbf{B}_0}, \quad \mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}, \quad \mathbf{B} = \mathbf{F}\mathbf{B}_0.$$

where $\mathbf{H}$ is the magnetic field.

The corresponding Lagrangian equations are the nominal stress $\mathbf{T} = \mathbf{F}^{-1}\boldsymbol{\tau}$ given by

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}$$

and the magnetic field

$$\mathbf{H}_0 = \frac{\partial \Omega}{\partial \mathbf{B}_0}.$$

Finally, the theory is applied to the solution of a prototype boundary-value problem in which a rectangular block is bent into a sector of a circular cylindrical tube with a referential magnetic field normal to one of the faces of the block. A closed-form solution for this problem is obtained for a particular choice of energy function. The magnetic field in the deformed configuration becomes radial and the stress/strain response stiffens with increasing magnetic field strength.

References

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