A NEW ROAD TO CHAOS IN DYNAMICAL SYSTEMS WITH IMPACT INTERACTIONS

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Summary A new bifurcation of dynamical systems with impact interactions is investigated in the report. It is shown how Smale's horseshoes can emerge in considered systems as a consequence of the bifurcation. A theorem is proved in the paper that existence of Smale's horseshoe implyies chaotic behavior of motions of the systems. A model of vibroimpact device is considered as an example. According to the computational investigations, stationary motions of the system are chaotic and have unusual limiting sets.

Recently there was placed high emphasis on investigation of roads to chaos. Owing to this fact, today a number of roads to chaos in smooth systems is known. They are: cascade of bifurcations of period doubling, bifurcation of quasi-periodic motions, bifurcation of systems with structurally unstable homoclinic curve, bifurcation of saddle-focus singularity in system with homoclinic curve and intermittency. As for piecewise smooth systems, earlier there was considered road to chaos connected with coming of the periodic motion to the boundary of smoothness breakdown of the systems.

In present report a new [1] road to chaos in dynamical systems with impact interactions is investigated. The under study road to chaos is connected with the bifurcation of the periodic motion. The bifurcation occurs when the periodic motion comes to the boundary of the region of infinite-impact motions existence. The term infinite-impact motion indicates [2] motion with infinitely many impact interactions in a finite time interval.

ANALYTIC INVESTIGATION

The following mathematical simulation [3] is considered for dynamical systems with impact interactions. A momentary impact interaction takes place on the hypersurface $x_n = 0$, on which phase variables $x_1, \dots, x_{n-1}$ change by jump (x_n remains equal to 0) according to formulas:

$$\begin{aligned} &x_1^+ = H_1(x_1^-, \dots, x_{n-1}^-) = x_1^- H_{11}(x_1^-, \dots, x_{n-1}^-), \\ &x_i^+ = H_i(x_1^-, \dots, x_{n-1}^-) = x_i^- + x_1^- H_{1i}(x_1^-, \dots, x_{n-1}^-), \\ &i = 2, 3, \dots, n-1. \end{aligned} \tag{1}$$

And for $x_n > 0$ change of the phase variables obeys differential equations

$$\begin{aligned} &dx_i / dt = \dot{x}_i = \Phi_i(x_1, \dots, x_n), \quad i = 1, 2, \dots, n-1, \\ &dx_n / dt = \dot{x}_n = x_1 \Phi_{n1}(x_1, \dots, x_n) + x_n \Phi_{nn}(x_1, \dots, x_n) = \Phi_n. \end{aligned} \tag{2}$$

Here $x_1^-, \dots, x_{n-1}^-$ and $x_1^+, \dots, x_{n-1}^+$ - are pre-impact and after-impact values of the variables, respectively; all stated functions are smooth of class C^m, $m \geq 3$, and t - time. The phase space of system (1) - (2) consists of points $(x_1, \dots, x_{n-1}, x_n \geq 0)$.

Suppose that system (1) - (2) have singularities defined by the conditions [4]: $x_n = 0$, $x_1 = 0$, $\ddot{x}_n = 0$, $\dddot{x}_n > 0$ and $x_n = 0$, $x_1 = 0$, $\ddot{x}_n = 0$, $\dddot{x}_n < 0$. On fig. 1 this singularities are denoted by points M_* and N_*, respectively. (Fig. 1 represents hypersurface $x_n = 0$.) Then infinitely many sets D_N, $N = 1, 2, 3, \dots,$ exist on the manifold $x_n = 0$, $x_1 \geq 0$. Phase trajectories starting at points of set D_N leave small neighborhood of the impact hypersurface $x_n = 0$ after N impact interactions.

Mapping $T = T_1 T_2$ [3] of manifold $x_n = 0$, $x_1 \geq 0$ into itself is used to describe behavior of phase trajectories starting at points of sets D_N. Here: T_1 maps point $(x_1 \geq 0, x_2, \dots, x_{n-1}, 0)$ into point $(x_1^- \leq 0, x_2^-, \dots, x_{n-1}^-, 0)$ by trajectories of system (2); and T_2 realizes impact interaction by formulas (1).

In the dynamical systems with impact interactions, as a consequence of the investigated bifurcation, Smale's horseshoes can emerge in the following way. Intersection of sets D_N and $T^{N+k} D_N$ (if the periodic motion comes to boundary γ_* after k impact interactions) forms Smale's horseshoe as it is depicted on fig. 1. Here γ_* is the boundary of the region of infinite-impact motions existence.

Earlier the term of Smale's horseshoe was used only for hyperbolic systems. As considered class of systems is not hyperbolic, the term of Smale's horseshoe is specified in the report.

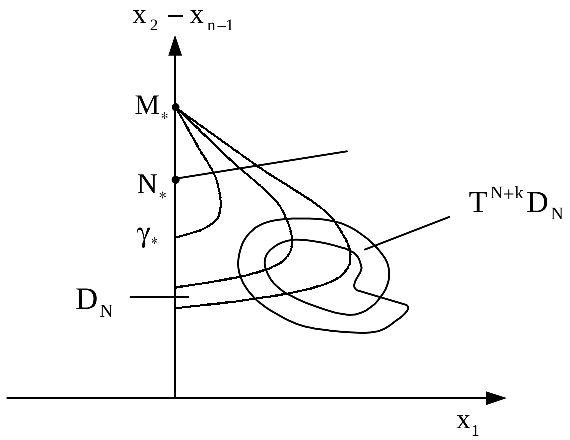

Figure 1

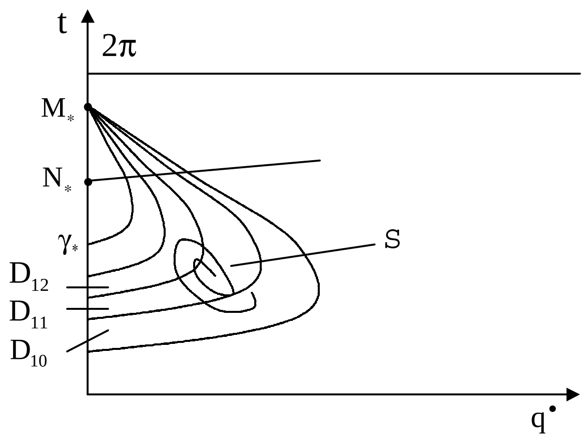

Figure 2

The following theorem is proved in the paper.

Suppose any Smale's horseshoe exists in system (1) - (2); then every infinite symbolic sequence $a_1, a_2, a_3, \ldots$ *corresponds to at least one trajectory of the system. Here variable* a_k, $k = 1, 2, 3, \ldots,$ *is equal to 0 or 1; and symbol "0" corresponds to hit of the trajectory into one part of Smale's horseshoe while symbol "1" corresponds to hit of the trajectory into another part of Smale's horseshoe.*

There's proved a consequence of the theorem that *existence of Smale's horseshoe implies presence of chaotic motions in the system.* Recall that chaotic motion is a motion having the following two properties: sensitivity of dependence from initial conditions and irregularity of temporary evolution.

COMPUTATIONAL INVESTIGATION

The following model of vibroimpact device is considered as an example

$$q^{\bullet\bullet} + \lambda^2 q = V \sin t + 1, \quad \text{for} \quad q > 0,$$
$$q^{\bullet +} = -Rq^{\bullet -}, \quad \text{for} \quad q = 0. \tag{3}$$

The phase space of system (3) consists of points $(q, q^{\bullet}, t)$ where $q \geq 0$, $-\infty < q^{\bullet} < +\infty$, $0 \leq t \leq 2\pi$. Points $(q, q^{\bullet}, t = 0)$ and $(q, q^{\bullet}, t = 2\pi)$ are supposed to be identical.

In computer simulation of system (3) values of parameters are determined when stationary motions are chaotic after examined bifurcation. In these cases Smale's horseshoes are observed also. The last fact explicates existence of chaotic motions in view of the proved theorem.

There are also discovered values of parameters when chaotic motions have unusual limiting sets after examined bifurcation. For example, at $V = 4$, $R = 0.68$, $\lambda = 2.55$ such unusual limiting set S is depicted on fig. 2. This means the following. Fig. 2 represents manifold $q = 0$, $q^{\bullet} \geq 0$ of the phase space of system (3). Suppose that P is any point of this manifold from given neighborhood of set S. Then motion of system (1) - (2) starting at point P is a chaotic motion. All points $G^k P$ belong to sufficiently small neighborhood of set S if number k is sufficiently large. Besides, it is true that $G(S) = S$. Here mapping G is defined thus: $GP = T^{N+1}P$ if $P \in D_N$.

CONCLUSIONS

A new bifurcation of the periodic motion in dynamical systems with impact interactions is researched. Emergence, after the bifurcation, of chaotic motions is explained by occurrence of Smale's horseshoes. In computer simulation of specified system different Smale's horseshoes and limiting sets are observed.

References

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[3] Gorbikov S.P.: Singularities of Phase Space of Dynamical Systems with Impact Interactions. *Mechanics of Solid*, 3, 23-26, Moscow 1987 (in Russian).

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