ATTRACTOR AND PATTERN CONTROL IN NONLINEAR MEDIA BY LOCALIZED DEFECTS

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Summary We consider pattern and attractor control in nonlinear dissipative systems. We develop an analytic approach to attractor control for neural, genetic networks systems of coupled oscillators and spatially extended systems. In particular, we apply this method for some systems of Ginzburg-Landau's type and others.

1. Introduction. In the last decade, a great attention has been given to chaos existence [1,18], pattern formation and control. Complicated patterns and large time behaviour can be observed in mechanics, chemistry, biology, physics (liquid crystals, magnetic thin films, Langmuir monolayers, polymers [2, 3]).

We propose new methods for pattern control in continuous and discrete nonlinear dissipative media. We first describe an analytic theory of the attractor and pattern control for the Hopfield neural networks. Basing on it, we then obtain algorithms of pattern and attractor control for many mechanical, physical and biological systems. In particular, we consider systems of Ginzburg-Landau's type, genetic networks, and some hydrodynamical models.

2. Attractor control for neural networks

Consider the Hopfield system (a basic model for so-called attractor neural networks)

$$\frac{dq_i}{dt} = \sigma(\sum_{j=1}^{m} K_{ij} q_j - \eta_i) - cq_i, \qquad (2.1)$$

where m is the number of neurons, the matrix K_{ij} defines the interaction between neurons, $c > 0$ is a constant, and η_i are thresholds. The functions $q_i(t)$ define neuron states. The main result is given by the following theorem [4,5]:

Theorem . *Suppose a system of n ordinary differential equations has a structurally stable invariant set $\mathcal{A}$. Then, for any n, parameters m, K, η, c can be adjusted in such a way that system (2.1) will have an invariant set $\mathcal{A}'$ topologically equivalent to $\mathcal{A}$. Dynamics of (2.1) restricted to $\mathcal{A}'$ is orbitally topologically equivalent to the given dynamics on $\mathcal{A}$. If $\mathcal{A}$ is an attractor, then $\mathcal{A}'$ also is an attractor for (2.1).*

This yields that any hyperbolic chaos (studied by works of S. Smale, D. Ruelle and R. Takens, D. V. Anosov et al., see [1,18]) can appear in the Hopfield dynamics. In particular, there are possible a complicated time behaviour connected with transverse homoclinic orbits. Structural stability means that, topologically, dymamics on invariant set persists under small C^1 perturbations (hyperbolic sets possess this property).

Moreover, the neurons form complicated coherent patterns. There are $n << m$ "leading" neurons (for instance, $q_1, ..., q_n$). The dynamics of the remaining ones is completely captured by these leading neurons. Namely, $q_i = \sum_{j=1}^{n} B_{ij} q_j$, where $i > n$ and $\mathbf{B}$ is some matrix. Similar results hold for networks with discrete time [6].

The proof is constructive and gives a method for the attractor control. It is based on multilayered and wavelet approximations. Notice that if $\mathcal{A}$ is the global attractor of a prescribed system of n differential equations, the paramaters in (2.1) can be found in such a way that $\mathcal{A}'$ will the global attractor of (2.1) as well. In a sense, we can "prescribe" dynamics (2.1) (locally, or globally).

If we remove the condition of structural stability, we can control families of trajectories of eq. (2.1) on large (but bounded) time intervals $[0, T]$, where T can be chosen arbitrarily. In this case the parameters in (2.1) depend on T and $m \to \infty$ as $T\infty$.

3. Attractor and pattern control for spatially extended systems.

These results are also useful in finding of other coupled oscillator and spatially extended systems with complicated large time behaviour.

We show that Theorem can be extended to a large class of coupled oscillator systems. We demonstrate that eq. (2.1) can serve as a normal form describing small oscillations at fixed points for many such systems.

It allows us to find an analytical approach on dynamics control for a number of important systems (however, it is necessary to note, that in contrast to (2.1), in general we cannot control the global attractor completely, i.e., we can control only small oscillations at some equilibrium states).

Consider now systems of partial differential equations describing nonlinear dissipative continuous media. Basic ideas are as follows. For such systems a complex behaviour is often generated by interface motion (for reaction-diffusion models) or vortex movement (for hydrodynamical models). We apply the following approach: motion of localized modes (interfaces, vortices etc.) can be effectively controlled by well localized inhomogeneities. The localized modes approach the space defects (pinning effect) and interact with other modes through these defects. This interaction is nonlocal (similarly to (2.1)) and can be described by coupled oscillator systems admitting chaotic behaviour. In this approach, the important mathematical tool is so-called method of realization of vector fields [17].

We have found many important systems, where complicated patterns or chaotic attractors exist and where these attractors and patterns are controllable by system parameters.

The first important example is given by genetic networks. These systems are intensively studied [12-16]. We have considered the model proposed by [16] since this model is similar to the Hopfield equations (2.1).

It can be shown analytically that this model are capable to generate any spatio-temporal patterns [9, 10]. Moreover, there is an numerically effective algorithm allowing us to adjust a gene interaction producing a given pattern. So, genes are capable to produce any time sequences of any space structures.
Another example is two dimensional system of Ginzburg-Landau's type [5,7,8]:

$$u_t = \epsilon^2 u - u^3 + a(x,y)u + b(x,y)v, \tag{3.1}$$

$$v_t = \Delta v - c^2 v + g(x,y)u. \tag{3.2}$$

We consider this system in a two-dimensional rectangle under zero Neumann conditions. Here control parameters are small inhomogeneities a, b and g and ϵ. For this problem our approach essentially uses ideas of [2,3]. The coefficients a, b and g can be found as sums of well localized peaks. The pattern is formed by a number of interfaces (kinks) parallel to y with coordinates $q_i(t), i = 1, ..., m$ depending on time. This solution describes multilayered patterns (which is well studied for the case $a, b, g = const$ [2,3]).
For interface coordinates one obtains a system of differential equations, which is a small perturbation of (2.1) for small ϵ and appropriate functions a, b and g. The matrix K and coefficients η_i depend on a, b and g. Changing these inhomogeneities we can obtain any parameters K, η. It allows us to show that there is possible a complicated large time behaviour [5] (but, for each time moment, the spatial structure is simple: it is always a multilayered pattern).
We have considered three component generalizations of system (3.1), (3.2) [8]. Numerical and asymptotical calculations show existence of more complicated space patterns.
Recently we have extended this method to vortex motion in two-dimensional hydrodynamical systems. We have found a linear feedback boundary control for two-dimensional vortex motion. These boundary control uses a permeable wall following ideas [11].

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