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DISTRIBUTED PARAMETER CONTROL OF A 2D ACOUSTIC HELMHOLTZ PROBLEM ON A HALFSPACE

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Summary In this work we present the formulation and numerical solution of a distributed parameter control problem for the acoustic equation in a halfspace with potential applications to the seismic insulation of surficial structures. We consider the case of SH-waves in a two-dimensional materially inhomogeneous halfspace. The goal is to invert for the necessary material injections in a pre-selected region near the free surface so that, for a range of excitation frequencies of an incoming disturbance, the displacement response on the free surface is constrained below a threshold value. We use an infinite-dimensional constrained optimization formulation, in which the constraints are given by a set of (uncoupled) Helmholtz problems corresponding to a range of excitation frequencies. The Helmholtz problems are discretized using a finite element formulation on a half disk with absorbing boundary conditions prescribed over the truncation boundary. We use a Lagrange-Newton-Krylov-Schur algorithm (LNKS) to solve the system of nonlinear PDEs that correspond to the first-order necessary optimality conditions. We present the formulation and numerical results.

INTRODUCTION

The desire to control the frequency content and the amplitude of the response to arbitrary excitations is commonly shared amongst various engineering applications. For example, highway noise-barriers aim at reducing the acoustic pressure signatures to predefined acceptable levels in residential areas by, essentially, altering the material profile that sound encounters in its path. Of interest here is a similar problem, also governed by the Helmholtz equation, that arises in the propagation of SH waves in soft soil deposits. We consider the case of a halfspace exposed to either traveling waves (antiplane case), or underground blasts. We explore the feasibility of limiting the response of a pre-selected near-surface region embedded within a valley of soft deposits, by artificially altering the deposits (material injections) within a narrow region lying between the origin of the excitation and the region of interest (Fig. 1).

Motivated by the physical application, we formulate a distributed control problem for the inhomogeneous Helmholtz problem. The objective is to minimize the amplitude of the response (displacement profile) on and near the surface (target zone Ω_t in Fig. 1). The control parameters are provided by the material properties (shear modulus μ) within a control zone Ω_c. We choose a constrained optimization approach in which we are minimizing the amplitude of the response across a narrow band of the frequency spectrum reflecting the expected excitation frequency content. From an optimization perspective, the control problem is a formidable one, for the number of unknowns one needs to solve for at each frequency is at least equal to twice the nodal points of the spatial discretization (for a scalar unknown). Until recently such problems were computationally intractable. Recent algorithmic approaches that combine Newton-Krylov methods with appropriate preconditioners that take advantage of the structure of the underlying PDEs, allow the efficient solution of the nonlinear equations corresponding to the first order optimality conditions.

There is a considerable amount of work on inverse problems for inhomogeneous media. Most problems however are related to the recovery of the material properties, or obstacle detection by observing the far-field response (e.g. [5], [6]). There is also work in noise reduction for acoustics, but the work is concentrated more on shape optimization (e.g. [7]). Moreover there is little work on the algorithmic scalability of the numerical methods.

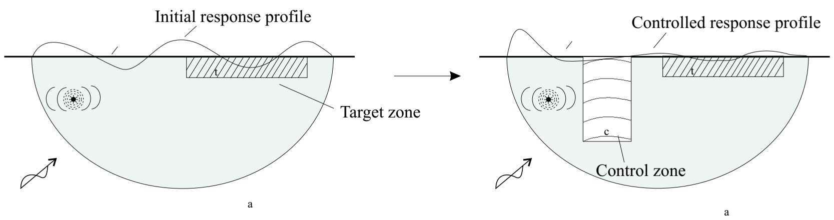

Figure 1. Schematic of a two-dimensional distributed control problem for SH waves (Helmholtz).

In short: We use a LNKS algorithm for the Helmholtz equation, we numerically investigate the feasibility of insulating surficial structures by soil injections, we incorporate multiple frequencies, we use second-order absorbing boundary conditions for the forward and adjoint operators, and we investigate the effect of different regularization functionals.

FORMULATION

We consider the halfspace Helmholtz problem for an inhomogeneous medium in the absence of damping. For simplicity, we assume constant density throughout the domain, but variable shear modulus (μ_0). In the following u denotes the displacement and u_ν its (outward) normal derivative along Γ_a and γ. We use $\mu(\mathbf{x})$ ($\mathbf{x} \in \Omega$) to denote a perturbation from the background material $\mu_0(\mathbf{x})$. We assume a smaller region, Ω_c, in which μ can be controlled, i.e. Ω_c is the support of $\mu(\mathbf{x})$. We are interested in the behavior of the system in m excitation frequencies ω_j. We introduce the following notation: $a_j(\mu, u, \tilde{u}) := -\int_\Omega (\mu_0 + \mu)\nabla u \cdot \nabla \tilde{u}\, d\Omega + \int_\Omega \omega_j^2\, u\, \tilde{u}\, d\Omega, \quad g(u, \tilde{u}) := \int_{\Gamma_a} \mu_0\, u\, \tilde{u}\, d\Gamma_a, \quad (u, \tilde{u})_D := \int_D u\, \tilde{u}\, dD$, where D denotes a generic integration domain. We define H^1 to be the space of functions with square-integrable first derivatives in Ω, and L^2 the space of functions that are square-integrable in Ω. Then, given an excitation $b_j(\mathbf{x})$ in L^2 at frequency ω_j a weak-form statement of the acoustic problem is to find $u_j \in H^1$ such that $a_j(\mu, u_j, \tilde{u}) + g(u_{j_\nu}, \tilde{u}) = (b_j, \tilde{u})_\Omega, \quad \forall \tilde{u} \in H^1$, were u_j is the solution that corresponds to the frequency ω_j. We use a FEM-based approximation scheme, and thus a truncated domain is necessary. On its boundary Γ_a an absorbing boundary condition has to be prescribed. The simplest one is given by $\mu_0 u_{j_\nu} - i\omega u_j = 0$. In our implementation we prescribe second-order boundary conditions; details can be found in [2].

Control problem: Using the above relations we seek to minimize a measure of the response u for several different frequencies. The objective functional is based on the observation of the displacement on the target domain Ω_t. We use the (square of the) $L^2(\Omega_t)$ norm to measure the amplitude of the response. We choose m frequencies in which we want to test our control. In that sense we want to choose μ as the minimizer of $\frac{1}{2}\sum_j (u_j, u_j)_{\Omega_t} + \frac{1}{2}(\nabla\mu, \nabla\mu)_{\Omega_c}$. The last term is a regularization term that controls the smoothness of the perturbation. Depending on Ω_c this term might be necessary for well-posedness. (Different regularization functionals, like total variation or L^2, are also possible.) For simplicity we explain our approach when $m = 1$. We introduce the adjoint λ, also in H^1, and we look for stationary points of the Lagrangian defined by:

$$\mathcal{L}(u, \lambda, \mu) := \frac{1}{2}(u, u)_{\Omega_t} + \frac{1}{2}(\nabla\mu, \nabla\mu)_{\Omega_c} + a(\mu, u, \lambda) + g(u_\nu, \lambda) - (b, \lambda)_\Omega. \tag{1}$$

Let $K := \{\mu \in H^1(\Omega_c) : \mu(\overline{\Omega_c}\backslash\gamma) = 0\}$. Taking variations with respect to u, λ and μ we obtain the so-called Karush-Kuhn-Tucker (KKT) first-order optimality conditions in which we seek u, λ in H^1 and μ in K such that:

$$\begin{aligned} a(\mu, u, \tilde{u}) + g(u_\nu, \tilde{u}) - (b, \tilde{u})_\Omega &= 0, \quad \forall \tilde{u} \in H^1 && \text{forward problem,} \\ a(\mu, \lambda, \tilde{\lambda}) + g(\lambda_\nu, \tilde{\lambda}) + (u, \tilde{\lambda})_{\Omega_t} &= 0, \quad \forall \tilde{\lambda} \in H^1 && \text{adjoint problem,} \\ (\nabla\mu, \nabla\tilde{\mu})_{\Omega_c} + a(\tilde{\mu}, u, \lambda) &= 0, \quad \forall \tilde{\mu} \in K && \text{control problem.} \end{aligned} \tag{2}$$

It is easy to show that the boundary condition for the adjoint variables is the same with the one of the forward problem. Finally, an additional constraint of the form $\mu \geq -\alpha\mu_0$ is required; $0 < \alpha < 1$ is a parameter chosen to keep $\mu_0 + \mu$ positive and thus ensure strong ellipticity. This constraint is treated with a penalty term in the objective function.

NUMERICAL SOLUTION

One way to solve (2) is to use a nonlinear Gauss-Seidel iteration: given some initial μ_0 we solve the forward problem for the displacement field, then we use this field as a forcing term for the adjoint equation and finally we solve the control equation for an update μ. This is a popular approach since it is relatively easy to implement, especially for self-adjoint problems. However this approach is expensive, since at each optimization step we solve $m \times 2$ Helmholtz problems (for the forward and adjoint problem) plus one Laplacian in Ω_c for the control equations. A similar approach is to first linearize (2) and then apply the Gauss-Seidel sweep. In the optimization community this method is also known as reduced space sequential quadratic programming (RSQP). Instead, here we use a Newton-Krylov method applied directly to the KKT conditions. The key component for the success of the method is the application of a preconditioner which is based on an approximate version of the RSQP step. The method is termed *Lagrange-Newton-Krylov-Schur* and it first appeared in the context of boundary flow control [3, 4]. Variants of this method have been applied in inverse wave propagation, in which the acoustic propagation is written in the time domain [1]. (For inverse problems phase information is important and time-domain formulations are preferable.) The forward, adjoint, and control equations are discretized using Galerkin FEM with quadratic elements. Results will be presented on the well-posedness of the optimization problem, with a discussion on the numerical and scalability results for different mesh sizes, loading scenarios, and regularization functionals.

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