DIRICHLET/DIRICHLET AND NEUMANN/NEUMANN PARALLEL NON-OVERLAPPING DOMAIN DECOMPOSITION METHOD

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Summary Solution of two-dimensional Helmholtz equation using spectral non-overlapping domain decomposition method on parallel computer is presented. The Chebyshev tau method is used for discretization of the subdomian problems, and diagonalization technique is used for solution of the local system of equations.

The paper presents the tests of the algorithms for parallel solution of two-dimensional Helmholtz equation using non-overlapping spectral domain decomposition method. The 2-D Helmholtz equation is shown below with Dirichlet boundary conditions at the bounds $\partial\Omega$ of computational domain Ω

$$\nabla^2 u - \sigma u = f(x,y) \quad \text{in} \quad \Omega, \qquad\qquad u = g \quad \text{on} \quad \partial\Omega \tag{1}$$

where σ is the constant value, $f(x,y)$ is given function of the space coordinates and Ω is a square $(-1,1)\times(-1,1)$.
The computational domain Ω was split following one direction into the N_{el} number of subdomains. For all of the tests presented here the number of processors was equal to the number of subdomains. The Helmholtz equation (1) was transformed into the local coordinate systems and continuity of the solution at the interfaces was enforced specifying the suitable patching conditions. As it was proposed by Orszag [1] for solution of the second-order differential equation the suitable patching conditions at the interfaces are continuity of the function and continuity of its first-order derivative.
Solution of two-dimensional Helmholtz equation with Dirichlet as well as Neumann boundary conditions is presented using the iterative method proposed by Zanolli [5] and its modification proposed by Louchart and Randriamampianina [3]. The novel feature presented in the paper consists of the solution of the Helmholtz equation using the original iterative method proposed in [6]. The parallel performance of this method was already compared in [6] with the efficiency obtained using the other iterative methods for the solution of one-dimensional Helmholtz equation. The present paper summarizes the tests performed for solution of two-dimensional problem using the iterative schemes presented above. The spectral Chebyshev tau method was used for discretization of subdomain problems. The local system of equations (at each of processors) was solved using the matrix diagonalization method proposed by Haidvogel and Zang [7]. In the present paper the diagonalization technique proposed by Haldenwang et al. [8] for the tau method was used. Diagonalization method allows for transformation of two-dimensional system of equations into the sequence of one-dimensional ones, reducing substantially the time needed for its solution. The computation of eigenvalues, eigenvectors and inversion of the coefficient matrix was performed once at the preprocessing step. Thus, at each iteration step the solution of the subdomain problem was obtained by matrix product. The results obtained using Zanolli, Louchart and Randriamampianina algorithms as well as the proposed method for solution of linear 2-D Helmholtz equation can easily be compared with the exact solution.
The research presented is aimed at development of new, effective solutions methods for Navier-Stokes (N-S) equations, however one should remember that solution of N-S equations using projection method proposed e.g. by Hugues and Randriamampianina [4] consists of solution of the Helmholtz problems. If one uses the projection scheme presented above (for two dimensional problem), then two Helmholtz equations for determination of velocity components and two Poisson equations ($\sigma = 0$ in Helmholtz equation) for computations of pressure predictor and corrector are needed to be solved. The constant σ in the Helmholtz equation is inversely proportional to the time step if solution of unsteady Navier-Stokes equations is considered. As the next stage, solution of the Navier-Stokes equations using the Hugues and Randriamampianina scheme is considered, with explicit treating of the non-linear terms.
The first of the iterative algorithms (Zanolli scheme) consists of solution of two subdomain problems. In order to fulfil the patching conditions at the interface the data transmission between subdomains is needed. The iterative step (i) consists of two stages. At the first stage the problem with Dirichlet boundary condition at the interface Γ is solved:

$$\begin{aligned} L u_{[1]}^{(i+1)} &= f_{[1]}(x,y) \quad \text{in} \quad \Omega_1 \\ u_{[1]}^{(i+1)}(a) &= g_- \\ u_{[1]}^{(i+1)}(\Gamma) &= B^{(i+1)} \end{aligned} \tag{2}$$

next the value of the first derivative at the interface is computed. The value of the first derivative is sent to the other subdomain and at the second stage of iterative step the problem with Neumann boundary condition at the interface is solved at the second subdomain:

$$
\begin{aligned}
Lu_{[2]}^{(i+1)} &= f_{[2]}(x,y) \quad \text{in} \quad \Omega_2 \\
u_{[2]}^{(i+1)}(c) &= g_+ \\
\frac{\partial u_{[2]}^{(i+1)}}{\partial n}(\Gamma) &= \frac{\partial u_{[1]}^{(i+1)}}{\partial n}(\Gamma)
\end{aligned} \tag{3}
$$

where the n is the unit vector normal to Γ. The value of the function B^{i+1} at the interface Γ between subdomains at each of iterative steps is computed using the following relation

$$
B^{(i+1)} = \theta u_{[2]}^{(i)}(\Gamma) + (1-\theta)B^{(i)}, \quad i \geq 1 \tag{4}
$$

where $u_{[2]}$ is the function value computed at the boundary of the adjacent subdomain, $B^{(i)}$ is the value of the function from the previous iterative step and θ is the relaxation parameter. Funaro, Quarteroni and Zanolli [2] showed that this algorithm converge in two iterations if computational domain is split into two subdomains for one- as well as two-dimensional problems. As it was mentioned above, the efficiency of Louchart and Randriamampianina algorithm was also considered. Using the Louchart and Randriamampianina scheme one may avoid synchronisation problem encountered in the original Zanolli method, where one of the processors must wait for the other at each stage of iterative step (Eqs. 2-3). In the modified Zanolli scheme, the Dirichlet/Dirichlet and Neumann/Neumann boundary conditions are set at the interfaces in the successive stages of iterative step, allowing more efficient parallel computing.

As it was shown in [3] solution of differential equation using iterative domain decomposition algorithms with Dirichlet/Dirichlet and Neumann/Neumann boundary conditions specified at the interfaces is effective if the number of the subdomains is small enough. However, when increasing the number of subdomains, substantial increase of the number of iterations needed to obtain converged solution was observed. As an alternative, a new iterative algorithm for parallelization of numerical code using Chebyshev tau method was proposed [6]. In the algorithm proposed the Dirichlet boundary conditions are specified at the interfaces. At the beginning of iterative process the values of the functions at the interfaces are set to arbitrary values. Next, the correction of the previous solution is computed at one of the processors by solution of the common problem taking into account the patching conditions at all of the interfaces considered. The patching conditions are: continuity of the function and its first-order derivative. Correction of the solution is scattered to all processors (subdomains) and the values of the functions at the internal boundaries are modified. The iterative process is continued until converged solution is obtained.

For all of the iterative methods considered, influence of the choice of the number of subdomains, relaxation parameter θ and constant σ in Helmholtz equation as well as influence of the type of boundary condition (Dirichlet or Neumann) at the boundaries $\partial\Omega$ of domain Ω on the convergence of iterative process was tested. The parallel performance of the iterative methods was measured by means of speed-up $S = t_1/t_p$, which is defined as the ratio of time needed to execute the program on one processor to the time needed by p processors.

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