COMPUTATION OF VISCOUS VORTICES WITH FULLY MESHLESS METHOD

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Summary Viscous vortex interaction is hard to compute due to numerical diffusion of traditional CFD methods. To tackle this problem, a Lagrangian vortex particle method that is fully mesh-less has been developed. Spatial adaption of the particles is provided using radial basis function (RBF) interpolation. The calculation of a quasi-steady vortex tripole has been used as proof of concept. The method has been implemented in parallel using the PETSc library, and the RBF system has been solved using pre-conditioned GMRES.

INTRODUCTION

The computation of viscous vortex flows at high Reynolds numbers using traditional numerical methods, such as finite elements or finite volume formulations, can be severely hindered by numerical diffusion. In certain applications, such as the interaction of vortices with structures, the useful prediction of loads on the structures depends on being able to resolve the concentrated areas of vorticity with accuracy, and with minimal numerical diffusion. Mesh-based methods, however, can never be devoid of numerical diffusion, and vortices are simply diffused too fast.

An alternative to traditional mesh-based methods for solving the Navier-Stokes equations is the Lagrangian vortex method. In the vortex method, elements of vorticity move with the local fluid velocity and thus satisfy the convective part of the vorticity transport equation without numerical diffusion. In other words, the nonlinear term of the Navier-Stokes equation is accounted for by integrating the ordinary differential equations for the particle trajectories, such that the truncation error is non-diffusive. The velocity field, in turn, is obtained from the vorticity via the Biot-Savart law, and the viscous effects can be provided by a variety of methods. This is an essentially mesh-free method; in practice, however, many workers rely on mesh-based additions to the method to provide for long-term accuracy. For example, the prevalent viscous scheme for vortex methods, called particle strength exchange or PSE, is derived from an integral operator that approximates the Laplacian in the viscous term. In this formulation, quadratures used to discretize the integral operator are very sensitive to the location of the vortex particles. Hence, workers apply a "remeshing" scheme, in which the vortex particles are periodically re-located onto a Cartesian mesh, and their circulation strength is interpolated to these locations using high-order kernels in tensor product formulations. See, for example, [1].

Recent numerical experiments have demonstrated, however, that standard remeshing schemes, although allowing long-time calculations, do introduce noticeable interpolation errors [2]. In fact, these schemes bring back the mesh to an otherwise mesh-less method, and with it, numerical diffusion.

A less ubiquitous viscous scheme than PSE, the core spreading method, does not suffer from quadrature errors. On the other hand, some means of controlling particle core sizes is necessary to keep convection error small thereby ensuring convergence, tackling the concerns of [3]. Application of vortex splitting has been proposed for this effect, and proved to be convergent [4]. However, vortex splitting, like remeshing, introduces numerical diffusion.

In this work, a mesh-free method for processing the locations of vortices in the Lagrangian formulation, as well as controlling core sizes when core spreading is used to provide viscous effects, is formulated and demonstrated. This method is based on radial basis function (RBF) interpolation ideas. Proof-of-concept has been carried out by computing the tripole attractor of a quadrupole-perturbed monopole. Work in progress includes implementation of the method in parallel using the PETSc library [5], and the computation in parallel of the interaction of viscous vortices.

NEED FOR SPATIAL ADAPTION AND MESH-LESS FORMULATION

Regardless of whether the vortex method is implemented with an integral operator-based viscous scheme, or any other (*e.g.*, core spreading, diffusion velocity, taking the Laplacian of the basis function), there is a need for providing some form of spatial adaption. This is due to the fact that the cores of the vortex particles, or 'blobs', must have some overlap to be able to represent accurately the smooth vorticity field. Thus, gaps in the particle distribution produced by the flow strain in a time-marching calculation will degrade the accuracy. Recent numerical experiments demonstrate how the accuracy of the blob discretization deteriorates over several orders of magnitude when the overlap ratio, defined as the inter-particle spacing divided by the core size (h/σ), increases passing through the value of one [2]. Using a new parallel implementation of the vortex blob discretization, the cited experiments have been replicated, and the results are presented in Figure 1. The test case is the classic Lamb-Oseen vortex, which was discretized using Gaussian vortex blobs on a square lattice; the blob circulation strengths were obtained by solving the RBF-like linear system, where the right-hand-side vector corresponds to the vorticity values at

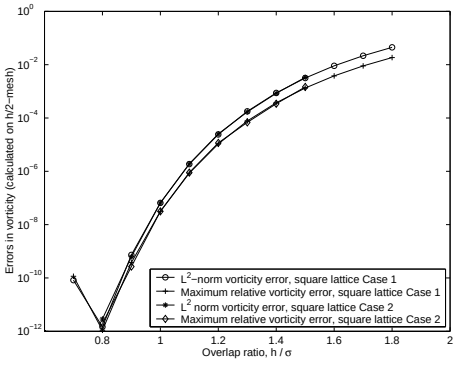

Figure 1. Blob discretization errors.

particle centres. That is, one solves for the γ_j's in $A_{ij}\gamma_j = \omega_i$ (indicial summation implied), with $\omega_i = \omega(x_i)$ and $A_{ij} = \frac{1}{2\pi\sigma^2}\exp\left\{-(x_i - x_j)^2/2\sigma^2\right\}$. This linear system was solved using preconditioned GMRES, a solution approach that was demonstrated for other radial basis functions (not Gaussian) in [6], using a specially designed pre-conditioner. Here, in contrast, the built-in pre-conditioners of the PETSc library were used, which can be specified as run-time options (four or six processors were used). In the figure, Case 1 corresponds to a set of discretizations using a fixed value $h = 0.01$ (about 10^4 particles) and variable core size σ; Case 2 corresponds to fixed $\sigma = 0.01$ and variable h, resulting in variable number of particles between 4500 and 15000. The vorticity error increases again at the lowest h/σ due probably to exceeding ill-conditioning of the linear system; indeed, at $h/\sigma = 0.8$ convergence of the GMRES at the desired tolerance was achieved with 59 iterations, whereas 1590 iterations were needed at $h/\sigma = 0.7$.

The "super-exponential" dependence on the overlap ratio of the vortex blob discretization demonstrated in Figure 1 justifies the need for spatial adaption in a Lagrangian method. Without it, as the particles are allowed to move with the flow, overlap may be lost in areas of the flow and accuracy would be degraded. To provide spatial adaption for the vortex method without resorting to a mesh-based approach, radial basis function (RBF) interpolation [7] is applied.

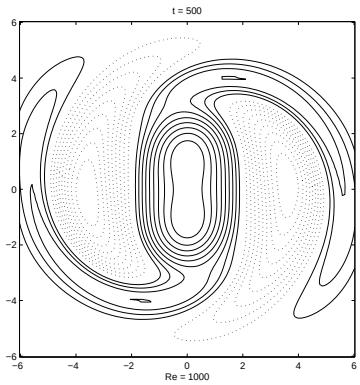

Figure 2. $Re = 10^3$.

Time-marching numerical experiments with the Lamb-Oseen vortex test case (using core spreading) have demonstrated that this approach can provide higher accuracy than standard remeshing [2]. In addition, the RBF spatial adaption allows one to correctly use core spreading for providing viscous effects, by controlling core size in the mesh-less adaption process. That is, core sizes are reset to their original smaller values during the RBF interpolation process. This offers an alternative to the vortex splitting method [4], which as already mentioned can generate considerable numerical diffusion.

Proof of concept numerical experiments have been performed using a Gaussian monopole with a quadrupole perturbation, which results in a quasi-steady state in the form of a tripole. The results of these calculations show noticeable improvement in comparison with similar calculations using core spreading with splitting [8], where numerical diffusion erodes the tripole attractor for the lower Reynolds number case (see Figure 2, which can be compared to Figure 8 –left– of [8]). For the higher Reynolds number (Figure 3), the present calculations reproduce the rotation angle and shape of contours of the tripole in the calculations of [8], but produce much smoother vorticity contours. Also, the decay of the quasi-steady tripole attractor in the $Re^{1/3}$ timescale is accurately obtained with the present method.

CONCLUSION

The computation of viscous vortex interaction with traditional CFD methods is hindered by numerical diffusion. To deal with this problem, a fully mesh-less method has been developed that is characterized by non-diffusive truncation errors. It consists of a new formulation of the vortex particle method, using core spreading for diffusion, and radial basis function interpolation to provide for spatial adaption of the particles and core size control. Numerical experiments indicate that this mesh-less formulation can provide higher accuracy in comparison with the standard remeshing that is used with the particle strength exchange viscous scheme, as well as the vortex splitting idea of [4]. Presently, we continue in the development of the parallel implementation using message passing with MPI and the PETSc library. An application is now being developed to study the viscous interaction of two co-rotating vortices, which develop an elliptical deformation of the cores in the weak regions of vorticity, which is very challenging to reproduce numerically. This problem was studied in [9] using spectral methods, but we estimate that the same accuracy can be obtained using the mesh-less vortex method with a considerably reduced problem size.

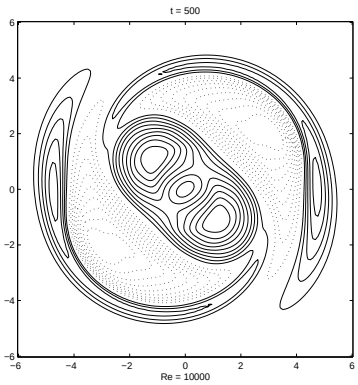

Figure 3. $Re = 10^4$.

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