

FERROFLUID MENISCUS SHAPE IN AN APPLIED UNIFORM HORIZONTAL OR VERTICAL MAGNETIC FIELD

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Summary The classic Landau and Lifschitz analysis [1] of the shape of a meniscus due to fluid wetting a vertical wall, including fluid weight and interfacial surface tension, is extended to ferrofluids stressed to magnetic saturation by a horizontal or vertical uniform magnetic field as shown in Figure 1. This analysis is the first step in a larger analysis to calculate the magnetic surface force due to ferrofluid surface shapes altered by magnetic fields and forces that cause surface driven flows in rotating magnetic fields [2].

Ferrofluid of density ρ in a vertical gravitational field, $\bar{g} = -g\bar{i}_z$, having magnetization \bar{M} in a magnetic field \bar{H} has pressure, gravitation, and magnetic forces related through the extended Bernoulli's equation [3]

$$p_b + \rho gz - \mu_0 \int_0^H M dH = \text{constant} \quad (1)$$

The i^{th} component of the interfacial surface force with interfacial normal \bar{n} is

$$(p_b - p_a)n_i + (T_{ija} - T_{ijb})n_i + T_{si} = 0 \quad (2)$$

where a and b are the coordinates just above and just below, respectively, of the interface, with interfacial deflection $\xi(x)$ as shown in Figure 1. The normal \bar{n} and surface tension force, \bar{T}_s , are given by

$$\bar{n} = \frac{\bar{i}_z - \frac{d\xi}{dx}\bar{i}_x}{\sqrt{1 + \left(\frac{d\xi}{dx}\right)^2}} \quad (3) \quad \bar{T}_s = -\frac{\sigma}{R}\bar{n} = -\sigma(\nabla \cdot \bar{n})\bar{n} = \frac{\sigma \bar{n} \frac{d^2\xi}{dx^2}}{\left[1 + \left(\frac{d\xi}{dx}\right)^2\right]^{3/2}} \quad (4)$$

where R is the radius of curvature at the interface with surface tension σ .

$$T_{ij} \text{ is the magnetic stress tensor } T_{ij} = H_i B_j - \frac{1}{2} \delta_{ij} \mu_0 H_k H_k \quad (5)$$

Using (3) – (5) in (1) and (2), with saturated ferrofluid magnetization M_s at angle θ_f from the horizontal, yields the governing differential equation

$$2\tilde{\xi}(\tilde{x}) \mp \frac{N(\sin^2 \theta_f - \cos^2 \theta_f) \left(\frac{d\tilde{\xi}}{d\tilde{x}}\right)^2 + 2N \sin \theta_f \cos \theta_f \frac{d\tilde{\xi}}{d\tilde{x}}}{\left[1 + \left(\frac{d\tilde{\xi}}{d\tilde{x}}\right)^2\right]} - \frac{\frac{d^2\tilde{\xi}}{d\tilde{x}^2}}{\left[1 + \left(\frac{d\tilde{\xi}}{d\tilde{x}}\right)^2\right]^{3/2}} = 0 \quad (6)$$

where non-dimensional variables are defined in terms of capillary length a

$$a = \left[\frac{2\sigma}{\rho g}\right]^{1/2}, \quad \tilde{\xi} = \frac{\xi}{a}, \quad \tilde{x} = \frac{x}{a}, \quad \tilde{z} = \frac{z}{a}, \quad N = \frac{1}{2} \frac{\mu_0 M_s^2 a}{\sigma} \quad (7)$$

For our experiments using Ferrotec EFH1 ferrofluid, $\mu_0 M_s \approx 0.0387$ Tesla, $\rho \approx 1169$ kg/m³, and $\sigma \approx 0.0258$ nt/m, so that $a \approx 2.1$ mm and $N \approx 49$. For horizontal ($\theta_f = 0$) or vertical ($\theta_f = \frac{\pi}{2}$) magnetic fields, (6) reduces to

$$2\tilde{\xi}(\tilde{x}) \mp \frac{N \left(\frac{d\tilde{\xi}}{d\tilde{x}}\right)^2}{\left[1 + \left(\frac{d\tilde{\xi}}{d\tilde{x}}\right)^2\right]} - \frac{\frac{d^2\tilde{\xi}}{d\tilde{x}^2}}{\left[1 + \left(\frac{d\tilde{\xi}}{d\tilde{x}}\right)^2\right]^{3/2}} = 0 \quad (8)$$

horizontal field ($\theta_f = 0$)

vertical field ($\theta_f = \pi/2$)

Defining $u = 1/[1 + (d\tilde{\xi}/d\tilde{x})^2]^{1/2}$ converts (8) to a form of Riccati's equation which can be solved by standard methods in terms of Airy functions Ai and Bi as

$$u(\tilde{\xi}) = \frac{2^{1/3}}{(N^{2/3})} \left[\frac{cAi'(\alpha) + Bi'(\alpha)}{cAi(\alpha) + Bi(\alpha)} \right], \quad \alpha = \frac{N(N \mp 2\tilde{\xi})}{(2N)^{2/3}} \quad (9)$$

where $Ai'(\alpha) = \frac{dAi(\alpha)}{d\alpha}$, $Bi'(\alpha) = \frac{dBi(\alpha)}{d\alpha}$, $c = -\frac{N^{2/3}Bi(\beta) + 2^{1/3}Bi'(\beta)}{N^{2/3}Ai(\beta) + 2^{1/3}Ai'(\beta)}$, $\beta = \left(\frac{N^2}{2}\right)^{2/3}$ (10)

The shape of the interface is governed by the differential equation $d\tilde{\xi}/d\tilde{x} = -[-1 + 1/u^2]^{1/2}$ (11)

with boundary condition $\frac{du}{d\tilde{\xi}} \Big|_{\tilde{\xi}=\tilde{h}} \mp N \cos^2 \theta_0 + 2\tilde{h} = 0$ (12)

where $\tilde{h} = \tilde{\xi}(\tilde{x}=0)$ is the non-dimensional fluid height, $\tilde{h} = h/a$, at the wall and θ_0 is the fluid contact angle at the vertical wall $d\tilde{\xi}/d\tilde{x}(\tilde{x}=0) = -\cot \theta_0$ (13)

as shown in Figure 1. With zero magnetic field, $N=0$, $\tilde{h} = \tilde{\xi}(\tilde{x}=0) = \sqrt{1 - \sin^2 \theta_0}$. The theory of (8) - (13) shows that horizontal magnetic field increases and vertical magnetic field decreases the meniscus height and width. Figures 2-3 show the theoretical meniscus height and shape for horizontal magnetic field.

Preliminary optical measurements of meniscus shape are shown in Figure 4 using measured narrow light beam reflections from the meniscus interface for various applied horizontal ($\theta_f = 0$) magnetic fields.

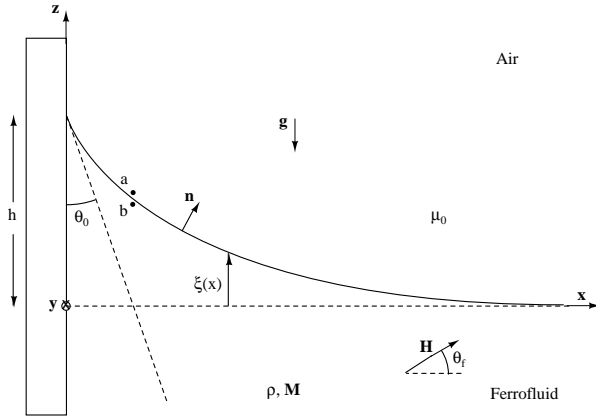


Figure 1 – A ferrofluid in a uniform applied magnetic field at an angle θ_f to the x-axis. The ferrofluid contacts a wall at $x=0$ at angle θ_0 rising to a height h . The meniscus has a shape $\zeta(x)$, where $\zeta(x=0) = h$.

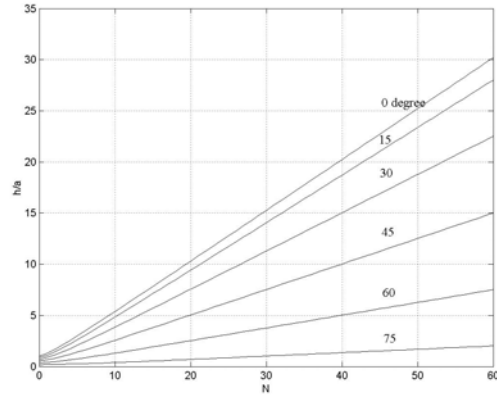


Figure 2 – Calculated non-dimensional height of fluid interface, h/a , at the $x=0$ wall as a function of parameter N for horizontal magnetic fields and various fluid contact angles θ_0 . The height h/a increases with increasing horizontal magnetic field.

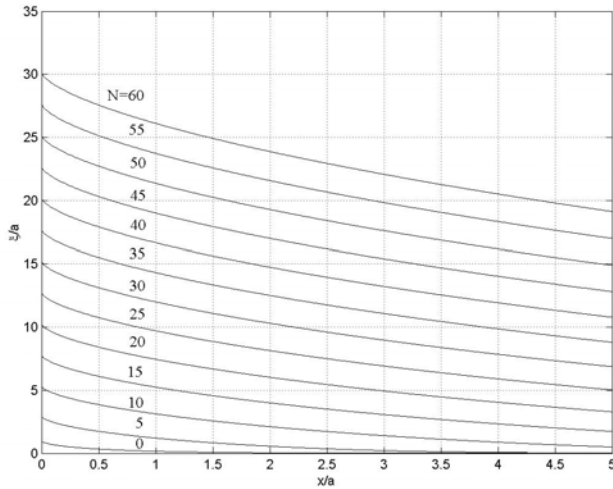


Figure 3 – Calculated non-dimensional interfacial deflection, ζ/a , as a function of non-dimensional distance from the wall, x/a , for various values of N , with a wall contact angle of $\theta_0 = 0^\circ$ in a horizontal magnetic field.

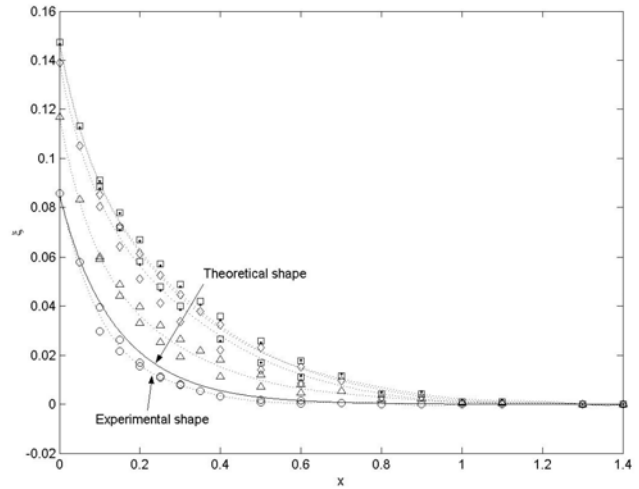


Figure 4 – Measurements, with least square fitting lines, of meniscus shape, $\zeta(x)$, for approximate magnetic fields of 0 (○), 70 (△), 130 (●), 190 (□), and 260 (◇) Gauss. The zero magnetic field theory agrees well with the zero magnetic field measurements for angle $\theta_0 = 57^\circ$.

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References

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