Breaking internal waves in a sheared fluid with critical layers.

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Summary This study is concerned with a new theoretical model for long internal waves in a sheared flow with a critical layer near either lower or upper rigid boundary. In such system even small disturbances will break and a zone filled with the mixed fluid will subsequently appear. One nonlinear differential equation for the wave amplitude is derived in the steady case, in which the nonlinearity arises essentially due to the presence of the mixed zone. Solutions of the obtained equation and their relevance to hydrophysical problems will be discussed.

MOTIVATION

Internal waves of permanent form owe their existence to a balance between nonlinear wave steepening and linear wave dispersion. For waves of small but finite amplitude nonlinearity is usually quadratic and the coefficient in front of this quadratic term is determined by the specific profiles of stratification and shear. Higher order expansions in wave amplitude may account for higher nonlinearities, however such approach is limited to waves that do not overturn. Overturning occurs at certain finite wave amplitude at which horizontal velocity approaches zero in a frame of reference moving with the wave. Above this critical amplitude a vortex core will be generated, which crucially alters the nature of the wave motion. The effect of the vortex core gives rise to an extra nonlinearity proportional to the 3/2 power of the difference between the wave amplitude and the critical amplitude [1]. Note that the form of this nonlinearity does not depend on the details of stratification and shear. However, in [1] no critical layers were assumed to exist and the ambient stratification and shear were assumed to be nearly uniform. On the contrary, this study addresses the case when the background shear is essentially non-uniform and it tends to be zero at the lower and upper boundaries thus critical layer inevitably exists for even small amplitude of the disturbance.

FORMULATION AND RESULTS

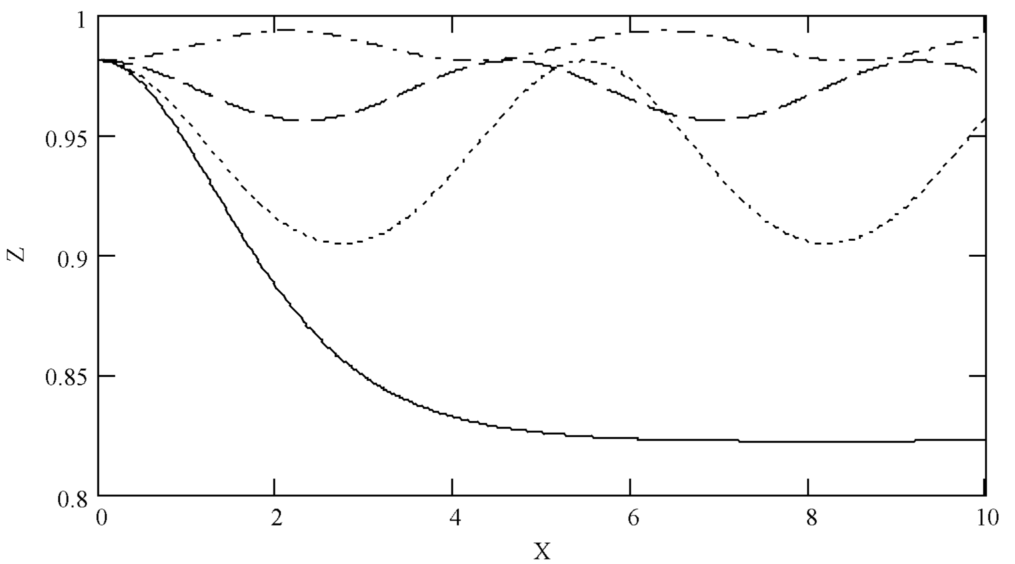
We consider the situation when the background shear is close to $\sin(\pi z / H)$, where z is the vertical coordinate and H denotes the total depth of the fluid. It can be shown that the Long's equation governing steady flow of an inviscid fluid between two rigid boundaries is linear in this case and thus it allows a separation of variables. Wave is assumed to be both fast and long enough so that the terms associated with the stratification and variations in the horizontal coordinate can be neglected in Long's equation in the lowest order. Compatibility conditions to the next order approximations yield the definition of the wave speed and the equation for the wave amplitude, where the first order terms are of the order of the deviation of the shear profile from $\sin(\pi z / H)$, which in turn is of the order of the effect due to the stratification. However, no assumption is made about the specific profile of stratification. The amplitude equation comes from the compatibility condition of the second order approximation,

$$A_{XX} = \pm\alpha_2 A^2 + \alpha_1 A + \alpha_3,$$

where X denotes the horizontal long wave coordinate, α_i are the constants derived from the integrals in the compatibility condition and α_2 is shown to be always positive. Nonlinear term in the amplitude equation appears solely due to the flow over the recirculation zone of the mixed fluid, which is assumed to be stagnant to the leading order in the frame of reference moving with the wave. Derivation of this term follows the procedure described in [1], however small shear in the vicinity of the mixed zone leads to a somewhat different relation between amplitude and profile of the boundary of the mixed zone, which is linear in this case. This leads to the quadratic nonlinearity instead of the 3/2-power law valid for a nearly uniform ambient shear [1]. The plus sign in front of the nonlinear term holds when $A > 0$, otherwise the minus sign should be taken. Positive amplitude corresponds to the case of the near surface location of the mixed zone. It is easy to reduce the obtained equation to the KdV type equation that however has the novel sign-changing nonlinearity. Since $\alpha_2 > 0$ the obtained equation has no solitary solutions for which the mixing zone vanishes at infinity, i.e. those with $A \to 0$ as $|X| \to \infty$. Moreover, α_1 should be positive to ensure the bounded solution, α_3 relates to the wave speed and should be in a certain interval for the same reason. All these properties are in marked contrast with the classical KdV. The maximum value of α_3 corresponds to the limiting state in the form of a semi-infinite expanding mixed layer (see the lowest solid curve in the figure). This solution also can be interpreted as a large solitary wave of elevation propagating on a new "thick" mixing layer. As α_3 decreases the disturbances become less intensive and shorter (compare solid, dotted, and dashed lines). That is the limiting state is a consequence of the broadening phenomenon quite typical for large internal waves [1,2]. It is interesting to note that the width of "moderate" disturbances changes slowly while their amplitude may vary quite dramatically (compare dotted and dashed lines in the figure). The similar effect has been observed experimentally in [3]. Further decrease of α_3 gives rise to disturbances that narrow the mixing layer (dash-dotted line). At a certain value of α_3 the length of the mixed zone located near surface becomes finite and another mixing (recirculation) zone appears near the bottom. So that the flow degenerates to series of independent mixing zones separated from each other and located alternatively near both boundaries. The minimum value of α_3 corresponds to another limiting state in which the mixing zone near the bottom becomes semi-infinite. Feasibility of such combinational flows is not clear at the moment and should be checked by direct computations. The situation when the induced mixing layer initially appears near the bottom has been considered similarly.

CONCLUSIONS

Finally, we conclude that a new model for long internal waves in a sheared flow with a critical layer near either lower or upper rigid boundary has been developed. In such system even small disturbances give rise to a formation of the zone filled with mixed fluid. One nonlinear differential equation for wave amplitude is derived in the steady case, in which the nonlinearity arises essentially due to the presence of the mixed zone. Solutions of the obtained equation include periodic waves on the induced mixed layer, the limiting state in the form of a semi-infinite expanding mixed layer and a combinational solution consisting of series of recirculation zones located alternatively near both boundaries.



Profiles of the near surface mixed layer for various wave amplitudes and speeds.

References

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