INTEGRATION OF THE EQUATIONS OF LONG WAVES IN THE CHANNELS AND JETS

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Summary. The Boussinesq approximation of the equations for long waves on the water surface in the channels and jets are considered. The equations are represented in the form of Lagrange. Lagrange function depends on a cross-section area and its derivative. For the steady-state motion in the general case we have formed three integrals with three arbitrary constants of integration. As a result we obtained the analytical solutions of many problems for cnoidal-type and solitude waves in the channels of different forms as well as in the jets taking into account surface tension or an elastic film.

PROBLEM FORMULATION

We consider problems of the propagation of waves on liquid surface in channels. The approximation where the ratio of the depth to wave length is assumed to be small and transverse acceleration of the liquid particles is neglected is called the shallow water approximation. The pressure in the shallow water approximation is determined according gydrostatics law. The next approximation in which the vertical acceleration is taken into account is called Boussinesq-type equations. Boussinesq equations usually imply two small parameters namely the ratio of depth to wave length and the ratio of the amplitude to depth. Boussinesq-type equations for the flow in channels [1] do not imply the latter parameter to be small. The cross-sectional area of the flow S, and the average (over this cross-section) velocity of the flow along the channel axis are taken to be the desired variables.

The equations are represented in the form of the shallow water equations.

$$\frac{\partial S}{\partial t} + \frac{\partial Su}{\partial x} = 0, \quad S\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x}, \tag{1}$$

where t is time, x is the coordinate along the channel axis, P is total pressure divided into density ρ of the liquid and S is cross section area of flow. Unlike the shallow water approximation, the Boussinesq approximation takes into account the transverse acceleration of liquid particles to determine the function P. The determination of the function P is based on the method of the generalized Lagrange coordinates. This method is applied to a transverse layer of liquid which has a small thickness l and cross-section area S. As a result, the quantity P is expressed in terms of the kinetic energy T and potential energy Π of the liquid layer depending on l, S. The thickness of a layer l is determined by the condition of constancy of the volume $V = Sl$.

In the present investigation [2] the equations (1) are generalized to the case of wave motion with allowance for the surface tension or an elastic film on the liquid surface. Also, the flow of liquid jets from a slot or round hole is considered, with the surface tension being taken into account.

RESULTS AND DISCUSSION

The function P explicitly expressed in term of the function S and its derivatives by the Lagrange method

$$P = S^2\left(\frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{S}} - \frac{\partial \Lambda}{\partial S}\right), \quad \Lambda = \frac{T - \Pi}{\rho l},$$

where $\Lambda(S, \dot{S})$ is the Lagrange function defined as the difference of the kinetic T and potential Π energy of the transverse motion at the section $x = const$.

For the steady-state motion, we have formed three integrals of the general system of equations. As a result, the system for the function $S(x)$ is reduced to the form

$$S'\frac{\partial \Lambda}{\partial S'} - \Lambda = \frac{Q^2}{2S} + A_1 + A_2 S. \tag{2}$$

The parameter Q is the flow rate equal to uS, A_1 and A_2 are arbitrary constants of integration. For certain values of Q, A_1, A_2 the integral defines solutions in the form of solitons. For periodic waves, these are expressed in terms of the wave characteristics, the amplitude, length, and average cross-sectional area. The relation between the flow rate Q and the wave parameter permits one to determine the velocity of the wave propagation. For the gravitational waves in a rectangular channel, we obtain familiar solutions in the form of periodic cnoidal waves and Boussinesq and Korteweg-de Vries solitons.

Examples 1. Capillary-gravitational solitude waves. A planar flow of an incompressible fluid is bounded, from one side, by the immovable bottom $z = 0$ and, from the other side, by the free surface $z = h(x)$ (x, z – are Cartesian coordinates). The shape of wave $h(x)$ can be determined by the equation (2). In terms of the dimensionless variable $X = x/h_\infty$ and $Y = h/h_\infty$, $h_\infty = h(\infty)$ the equation (2) is transformed to the following form

$$(Y')^2 = 3(Y-1)^2 \frac{c^2 - Y}{c^2 - \beta^2 Y}, \quad \beta^2 = \frac{3\sigma}{\rho g h_\infty^2}, \quad c^2 = \frac{u_\infty^2}{g h_\infty}, \quad Y' = \frac{dY}{dX},$$

where β^2 is the Bound number, c – dimensionless velocity of the wave propagation

The solution has the form of solitons and can be expressed in parametric form

$$\pm\sqrt{3}X = -\beta \ln\left|\frac{1+\beta t}{1-\beta t}\right| + B \ln\left|\frac{1+Bt}{1-Bt}\right|, \quad Y = c^2 \frac{1-t^2}{1-\beta^2 t^2}, \quad B = \sqrt{\frac{c^2-\beta^2}{c^2-1}},$$

where t is parameter $0 \le t < 1/B$.

We have: if $t = 0$ then $X = 0, Y = 1 + a = c^2$, where a is dimensionless amplitude, and if $t \to 1/B$ then $X \to \pm\infty, Y \to 1$. The soliton is symmetric and have smooth profile.

If $\beta < 1$, then $c^2 = 1 + a > 1$ the soliton amplitude is positive $(a > 0, ascent)$, whereas for $\beta > 1$: $c^2 = 1 + a < 1$ the amplitude is negative (inverted soliton). The inversion of the soliton when the Bound number exceeds a threshold value was established in the numerical investigations [3], [4]. The threshold value of the Bound number at which the solution amplitude changes in sign is determined analytically research in [5]. An analytical solution for inverse of soliton for high surface tension is obtained in [6].

Examples 2. Solitary waves in the triangular channel. It is shown, that in channels with rectangular or triangular cross-sections there exist waves in which the ascent of the free surface does not depend on transverse coordinate [1]. In this case, the system of Boussinesq equations has a simple exact solution. The shape of the wave for the channel of triangular cross section $z = p|y|$ (z, y are coordinates in cross section) is determined by the integral

$$\frac{1+3p^2}{3p^2}(Y')^2 = \left(1 - \frac{Y}{1+a}\right)(Y-1)^2 \left(1 + \frac{3+2a}{1+a}Y + \frac{4(1+a)}{3c^2}Y^2\right), \quad Y = \frac{h}{h_\infty}$$

$$c = \frac{1+a}{2+a}\sqrt{2 + 4a/3}.$$

In a similar way, one can obtain exact solution for the equations of periodic and solitary waves in channels and jets taking into account surface tension or an elastic film on the liquid surface, without assumption of the amplitude smallness.

CONCLUSIONS

The results of this work devoted to a novel analytical method of integration of the complicate Boussinesq equations for periodic and solitary waves in channels and jets taking into account different properties of the fluid surface are presented and discussed. The examples considered demonstraite that the method may be used to best advantage for analytical solution of many problems of wave dynamics.

References

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