

STANDING GRAVITY WAVES IN DEEP WATER

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Summary The two dimensional standing gravity wave problem, for an infinitely deep layer, is considered, based on the formulation of the problem as a second order non local PDE (see (1)). Despite the infinitely many resonances in the linearized problem we use the Nash-Moser implicit function theorem for proving the existence of standing waves corresponding to values of the amplitude ε having 0 as a Lebesgue point.

INTRODUCTION

We consider an infinitely deep layer of a perfect incompressible fluid, in two-dimensional potential motion under gravity, with a free surface without surface tension. We are interested in *solutions which are periodic in time and in the horizontal direction, and invariant under reflexion in the vertical axis*. The existence of solutions in the finite depth case were proved recently by Plotnikov and Toland in [5]. The problem we consider below has the additional difficulty to be infinitely degenerate at the linearized level, as noticed a long time ago (Poisson (1816), Cauchy (1827), Airy (1845), Stokes (1849)).

The formulation we take for the standing wave problem is the one introduced in Dyachenko et al [2], where w is a function of (x, t) , both even and 2π - periodic in x and t . In the following we denote by \mathcal{H} the spatial periodic Hilbert transform. The equation satisfied by w is a *second order nonlocal partial differential equation*, which may be written as follows (see [2] and [4])

$$\mathcal{F}(w, \mu) \stackrel{\text{def}}{=} \partial_t(L_{w'}\dot{w}) - \mu\mathcal{H}w' + \mathcal{H}\partial_x \left\{ \frac{1}{D}\mathcal{H}(L_{w'}\dot{w}\mathcal{H}L_{w'}\dot{w}) + (\mathcal{H}L_{w'}\dot{w})\mathcal{H}\left(\frac{1}{D}L_{w'}\dot{w}\right) \right\} = 0, \quad (1)$$

where a dot means a partial time (t) derivative, and a prime means a partial space (x) derivative ∂_x , and where

$$L_{w'}f = (1 + \mathcal{H}w')f - w'\mathcal{H}f, \quad D = (1 + \mathcal{H}w')^2 + w'^2,$$

and μ is the bifurcation parameter $\mu = gT^2/2\pi\lambda$, where T is the time period, λ is the spatial period, and g is the gravitational acceleration. With these equations, the free surface is given parametrically in physical coordinates (ξ, η) by

$$(\xi, \eta) = (x + \mathcal{H}w(x, t), -w(x, t)); \quad (x, t) \in \mathbb{R}^2.$$

The formulation (1) is equivalent to the classical formulation for sufficiently smooth solutions. We notice that $w = \text{const}$ is always a solution, and that we can look for solutions of (1) with zero average. Let us define for $m \geq 0$ suitable (Sobolev) spaces of doubly periodic functions

$$H_{\mathbb{H}\mathbb{H}}^{m,ee} = \{w \in H^m\{(\mathbb{R}/2\pi\mathbb{Z})^2\}; w \text{ is even in } x \text{ and in } t, w \text{ has zero average}\},$$

Then, for $m \geq 3$, \mathcal{F} is an analytic map from $H_{\mathbb{H}\mathbb{H}}^{m,ee} \times \mathbb{R}$ to $H_{\mathbb{H}\mathbb{H}}^{m-2,ee}$ and the linearization of (1) at the origin reads

$$\mathcal{L}_\mu u \stackrel{\text{def}}{=} D_w\mathcal{F}(0, \mu)u = \ddot{u} - \mu\mathcal{H}\partial_x u.$$

If μ is irrational, the kernel of \mathcal{L}_μ in $H_{\mathbb{H}\mathbb{H}}^{m,ee}$ is $\{0\}$. For $\mu = 1$ the kernel of \mathcal{L}_1 in $H_{\mathbb{H}\mathbb{H}}^{m,ee}$ is the *infinite-dimensional subspace* $E_0 \cap H_{\mathbb{H}\mathbb{H}}^{m,ee}$, with $E_0 = \text{span}\{A_q \cos q^2 x \cos qt; A_q \in \mathbb{R}, q \in \mathbb{N}\}$. For other rational values of μ , the infinite dimensional kernel of \mathcal{L}_μ is easily deduced from E_0 . For $f \in H_{\mathbb{H}\mathbb{H}}^{m,ee}$ which is orthogonal to E_0 in $L_{\mathbb{H}\mathbb{H}}^2$, there is a unique $u \in H_{\mathbb{H}\mathbb{H}}^{m,ee}$, orthogonal to E_0 in $L_{\mathbb{H}\mathbb{H}}^2$, which is solution of $\mathcal{L}_1 u = f$.

Since the pseudo-inverse of \mathcal{L}_1 is not regularizing and since the nonlinear terms in (1) lose 2 degrees of regularity, we need to adapt the Nash-Moser implicit function theorem for the bifurcation problem.

MAIN RESULTS

It is known that approximate solutions of the standing wave problem exist up to an arbitrary power of ε , where $\mu = 1 + \varepsilon^2/4$. More precisely we have the following

Theorem 1 *There are infinitely many approximate solutions w of (1) under the form of asymptotic expansions in powers of the amplitude ε of the wave*

$$w_\varepsilon^{(N)} = \sum_{1 \leq n \leq N} \varepsilon^n w^{(n)}, \quad \mu = 1 + \varepsilon^2/4, \quad w^{(1)} = \sum_{q \in I} \frac{\varepsilon_q}{q^2} \cos q^2 x \cos qt, \quad \varepsilon_q = \pm 1, \quad I \subset \mathbb{N},$$

with any finite subset I of \mathbb{N} , moreover $w_\varepsilon^{(N)}$ is such that $\mathcal{F}(w_\varepsilon^{(N)}, 1 + \varepsilon^2/4) = \varepsilon^{N+1} Q_\varepsilon$, where Q_ε is bounded in any $H_{\text{hh}}^{m,ee}$ when $\varepsilon \rightarrow 0$ (see [1] for $I = \{1\}$, [3] for the general result).

Our main result is the following

Theorem 2 *For m large enough, $N \geq 5$ and $\varepsilon_0 > 0$ small enough, there is a measurable set $\mathcal{E} \subset (0, \varepsilon_0)$, with 0 as a Lebesgue point, i.e.*

$$(1/r) \text{meas}\{\mathcal{E} \cap (0, r)\} \rightarrow 1 \text{ as } r \rightarrow 0, \quad (2)$$

such that for any $\varepsilon \in \mathcal{E}$ and $\mu = 1 + \varepsilon^2/4$, there exists a solution w of $\mathcal{F}(w, \mu) = 0$ in $H_{\text{hh}}^{m,ee}$, whose asymptotic expansion is $w_\varepsilon^{(N)}$, with $w^{(1)} = \cos x \cos t$ and $\|w - w_\varepsilon^{(N)}\|_{H^m} = O(\varepsilon^{N-1})$.

Remark 1: Same theorem holds when we take any rational value of μ , instead of 1.

Remark 2: We conjecture that theorem 2 also holds for any other type of asymptotic form $w_\varepsilon^{(N)}$ of the solution w , indicated at theorem 1. Notice that in this case the set \mathcal{E} of "good values" of ε may depend on the form of $w^{(1)}$.

SKETCH OF PROOF

Since there are infinitely many formal solutions, we need to specialize the solution we are looking for in searching for \underline{w} of order 1, defined by

$$w = w_\varepsilon^{(N)} + \varepsilon^{N-1} \underline{w}, \quad N \geq 5, \quad \mu = 1 + \varepsilon^2/4 \quad (3)$$

and the equation to solve is now

$$\mathcal{F}(w_\varepsilon^{(N)} + \varepsilon^{N-1} \underline{w}, 1 + \varepsilon^2/4) = 0. \quad (4)$$

The proof of Theorem 2 relies on the possibility to invert the linear operator \mathcal{A}_w (approximate linearized operator at a non zero point), defined for any u in $H_{\text{hh}}^{s,ee}$ by

$$\partial_w \mathcal{F}(w, 1 + \varepsilon^2/4) u = \mathcal{A}_w u + \Gamma(\mathcal{F}, u)$$

where $\Gamma(\mathcal{F}, u)$ cancels when w is a solution of $\mathcal{F}(w, 1 + \varepsilon^2/4) = 0$. The main result is that we are able to invert \mathcal{A}_w for \underline{w} bounded in $H_{\text{hh}}^{m,ee}$, $m \geq 14$, in controlling a *diophantine condition on ε* along the Newton iteration scheme used in the Nash-Moser theorem. This inverse operator satisfies

$$\|\mathcal{A}_w^{-1}\| \leq c/\varepsilon^2 \quad (5)$$

in the space $\mathcal{L}(H_{\text{hh}}^{s,ee}, H_{\text{hh}}^{s-2,ee})$, $s \leq m - 12$, which shows a loss of two derivatives and a bound which grows like ε^{-2} as $\varepsilon \rightarrow 0$. For obtaining such a result, suitable for proving theorem 2, we manage a succession of change of coordinates and change of variables in the spirit of the work [5], except the first change of coordinates which reduces all second order derivatives occurring in \mathcal{A}_w to a single second order derivative in t . The other key point is that we are able to invert explicitly the main order ε^2 of the "bifurcation part" of the linear equation $\mathcal{A}_w u = f$ in the infinite dimensional subspace $E_0 \cap H_{\text{hh}}^{m,ee}$, which introduces the factor $1/\varepsilon^2$ in the bound (5) of \mathcal{A}_w^{-1} , while the control of the diophantine condition leads to the loss of the two derivatives.

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