# MASS TRANSPORT DUE TO PARTIALLY REFLECTED WAVES IN A TWO-LAYER VISCOUS SYSTEM

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Summary Based on Lagrangian coordinates, a perturbation analysis is conducted to find the mean drifts due to partially reflected surface waves in a two-layer system, where the lower fluid is assumed to be much more viscous than the upper one. A single analytical expression is obtained for the mass transport velocity in each layer, incorporating all the cases where the wave can be progressive, standing or partially standing, and the domain can be closed or open at its far field.

## INTRODUCTION

We here study the mass transport due to long surface waves in a two-layer viscous system. An objective is to make clear the roles played by the free surface and interfacial set-ups in the mass transport problem, and to show the proper way of deriving these set-ups in the course of finding the mass transport velocity. Piedra-Cueva (1995) has presented a similar study on the mass transport in a two-layer system. He however used a questionable method to find the interfacial set-up. It has been shown by Ng (2004) that the Lagrangian expression for the set-up is not the same as the Eulerian counterpart. Apparently not aware of this difference, Piedra-Cueva (1995) has introduced an Eulerian equation for the Lagrangian set-up of the interface; this is obviously inconsistent. Furthermore, the present work is different from that of Piedra-Cueva (1995) in the following aspects. Piedra-Cueva (1995) considered only progressive waves, and found solutions separately for the two physical situations of bounded and unbounded domains. In this work, partially reflected surface waves are considered, with the pure progressive and standing waves being the two limiting cases. We derive for each fluid a single analytical expression for the mass transport velocity encompassing two possible (open and closed) far end conditions of the domain. To this end, the long-wave approximation is employed in this work to simplify the analysis. Use of a stream function is avoided, and expressions for the various set-ups are found more straightforwardly.

We also present a detailed numerical discussion, which is original in the literature, on the mass transport in the two layers when subject to changes in environmental factors like the wave reflection, the lower-layer fluid viscosity (such that the one-layer system is a limiting case), and the far-end condition of the domain. We shall in particular look into how the viscosity of the lower-layer fluid will affect the sense of rotation of the recirculation cells when under a standing wave, and the direction of the horizontal drifts when under a progressive wave in a closed system. The transition of the mean Lagrangian flow field from recirculation cells to unidirectional drifts when the reflection coefficient $R$ varies between 1 and 0 is also examined for a domain that can be closed or open. These features will distinguish the present work from other existing theoretical developments on mass transport in a two-layer system, such as Dore (1970, 1973), and Wen & Liu (1995).

## SOME RESULTS ON THE MEAN FLOW PATTERNS

The analysis is based on Lagrangian coordinates $(\alpha, \delta)$, which are the undisturbed horizontal/vertical positions of a fluid particle. The $\delta$-axis is directed vertically upwards with the origin $\delta = 0$ fixed at the equilibrium level of the interface. The variables shown below are normalized and distinguished by a caret. In the limiting case of a pure standing wave ($R$ = 1.0), the mass transport is known to exhibit a periodic cellular structure with a circulation cell formed over a quarter of the wavelength between a node and an antinode of the free surface. Figure 1 shows how the rotation of the cells formed in the two layers will vary with the viscosity of the lower-layer fluid. Plotted in this graph are the maximum mass transport velocities in the left cells, which occur immediately above and below the interface in the water core region and in the mud layer mid-way between a node and an antinode. These mass transport velocities give not only the strength of the circulation, and also the direction of rotation of the cells in the two layers. The flow patterns can be broadly divided into three cases. Case I is when the lower fluid is so highly viscous that it hardly flows under the wave excitation. The lower layer becomes virtually a rigid bed when $\hat{s}_m \equiv s_m / h > 0.5$ ($s_m$ is the Stokes boundary layer thickness in the lower fluid, and $h$ is the total fluid depth), and the classical case of a single layer (Longuet-Higgins 1953) is then recovered. In this limiting case of a single fluid, the left cell rotates clockwise, and a thin jet shoots upwards under an antinode. The flow in the lower layer is not appreciable until $\hat{s}_m < 0.2$. On reaching this stage, an anticlockwise cell forms on the left side in the lower layer, and a thin jet shoots downwards under an antinode in the lower fluid. It is remarkable that decreasing the viscosity of the lower fluid not only diminishes the mass transport velocity of the upper fluid, but also eventually reverses the rotation of the cells in the upper layer. This happens in this particular example when $\hat{s}_m = 0.142$. It is interesting to note that at this threshold point the mass transport field vanishes everywhere in the core region of the upper layer under a pure standing wave. In case II, when $\hat{s}_m$ is smaller than the threshold value, the left cell in either layer is anticlockwise, resulting in thin vertical jets shooting upwards now under a node in both fluids. On further decreasing the lower-fluid viscosity so that $\hat{s}_m$ then becomes only a small fraction of the fluid depth, a boundary layer structure also emerges near the bottom of the lower layer. In case III, when $\hat{s}_m < 0.035$, two distinct cells, one on top of the other, now appear between a node and an antinode in the lower layer.

We next examine the other limiting case, namely a pure progressive wave ($R$ = 0.0), when the system is closed. The mass transport is strictly horizontal in a progressive wave, but because of the return current the velocity will turn from positive to negative across part of a section in order to maintain a zero total flow in each layer. Figure 2 shows the dependence on $\hat{s}_m$ of the interfacial mass transport velocity and the mass transport velocity on the water free surface. We may infer from the sign of these velocities the change in direction of the drift profiles across the two layers. As shown in the figure, three possible cases corresponding to different ranges of $\hat{s}_m$ may again be classified. Case I again covers the case when the lower layer is virtually rigid, for which the classical behaviors for a long wave (Longuet-Higgins 1953), namely a strong forward velocity near the bottom and a milder backward velocity near the free surface, are exhibited in the upper layer. Decreasing the lower-layer viscosity will tend to decrease/increase the velocity of flow in the upper/lower layer. In the lower layer, the drifts near the interface and near the bottom will reverse their directions as $\hat{s}_m$ crosses the value of 0.075. It is remarkable that in case II the drift is backward immediately below the interface in the lower layer, but is forward near the interface in the upper layer. This implies a strong velocity gradient in the boundary layer above the interface. The flow in the upper layer is further weakened until it becomes completely quiescent when $\hat{s}_m$ decreases to the value of 0.035. This upper bound value for the present case III happens to be exactly the same as the one for the previous case III discussed above. On crossing over to case III, while the flow remains weak, the drift in the upper layer changes to become forward near the free surface and backward near the interface.

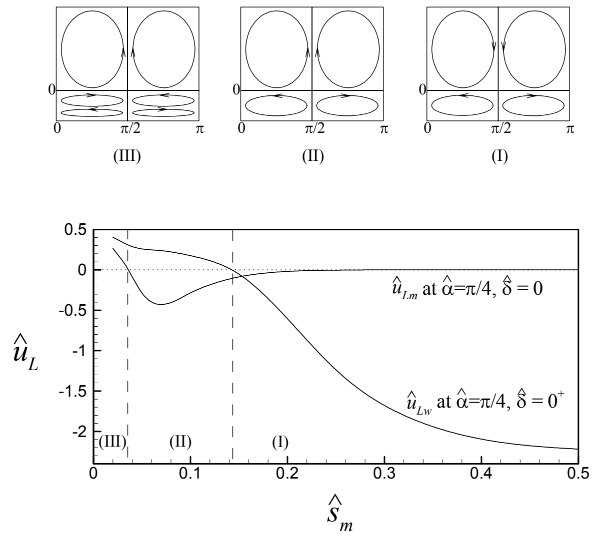


**Figure 1**. The maximum mass transport velocity in the upper core region $\hat{u}_{Lw}$ and that in the lower layer $\hat{u}_{Lm}$, which occur mid-way between a node and an antinode immediately above and below the interface, as functions of the mud Stokes boundary layer thickness $\hat{s}_m$ when under a pure standing wave ($R$=1.0).

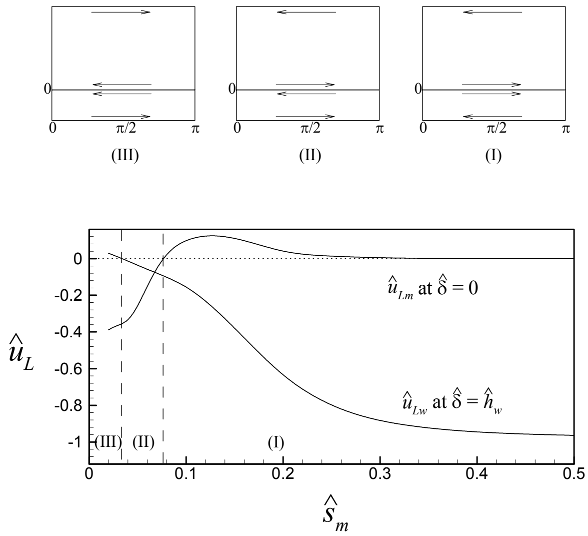


**Figure 2**. The mass transport velocity of the upper fluid at the free surface $(\hat{\delta} = \hat{h}_w)$, and that of the lower fluid at the interface $(\hat{\delta} = 0)$ as functions of the mud Stokes boundary layer thickness $\hat{s}_m$ when under a pure progressive wave in a closed system ($R$=0.0).

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