A MODEL FOR THE FORMATION OF 'OPTIMAL' VORTEX RINGS WITH TAKING INTO ACCOUNT VISCOSITY

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Summary The paper presents the model for formation of an axisymmetric vortex ring based on the viscous analogy to the Norbury vortices. By using the matching procedure suggested earlier and the obtained properties of the viscous vortex ring, it is found that when the limiting stroke value L/D ('formation number') reaches the value 4.0, the appropriate limiting values of the normalized energy and circulation become around 0.3 and 2.0, respectively. The predicted values are in agreement with the experimental data. An approach that enables to predict the 'formation number' is proposed.

INTRODUCTION

Vortex ring formation process was reproduced many times in the laboratory experiments Relatively recently, much attention has been paid to a structures produced for large piston stroke to the diameter ratios, L/D. The experiments by Gharib *et al.* [1] have shown that in this case the generated flow consists of the leading vortex ring followed by a trailing jet. The vorticity field of the leading vortex ring is disconnected from the trailing jet at a critical value of L/D (Gharib *et al.* refer to this as the "formation number"), at which time the vortex ring attains maximum volume, circulation and energy. The limiting value of the "formation number" follows from the direct experimental measurements by Gharib *et al*. who reported the values close to L/D=4 with a range of 3.6 to 4.5 for a variety of exit diameters, exit plane geometries, and non-impulsive piston velocities. Also the limiting values of the normalized energy and circulations were found experimentally as 0.3 and 2.0, respectively. The predictive models of the "optimal " vortex rings formation (Mohseni and Gharib [2]and Linden and Turner [3]) were developed by equating the motion invariants in the idealized slug-flow model with the corresponding properties of the Norbury vortices. However, real physical vortex rings have peakier vorticity distributions in contrast with the uniform vorticity density of the Norbury vortices. The model for the prediction of the optimal vortex ring formation on the base of the viscous analogy to Norbury vortices is discussed in this paper.

THE DIMENSIONLESS ENERGY AND CIRCULATION

The expressions for the circulation Γ_m, kinetic energy E_m and translation velocity U_m of the viscous vortex ring are given by (Kaplanski and Rudi[4])

$$\Gamma_m = \frac{M}{\pi R_0^2}\left(1-\exp\left(-\tau^2\right)\right),\ E_m = \frac{M^2\rho\tau}{2\pi^2 R_0^3}\left(\frac{1}{12}(\pi)^{1/2}\tau^2\ {}_2F_2\left[\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},3\right\},-\tau^2\right]\right),$$

$$U_m = \frac{M\tau}{4\pi^2 R_0^3}\left\{3(\pi)^{1/2}\exp\left(-\frac{\tau^2}{2}\right)I_1\left(\frac{\tau^2}{2}\right)+\frac{1}{12}(\pi)^{1/2}\tau^2\ {}_2F_2\left[\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},3\right\},-\tau^2\right]-\frac{3(\pi)^{1/2}}{5}\tau^2\ {}_2F_2\left[\left\{\frac{3}{2},\frac{5}{2}\right\},\left\{2,\frac{7}{2}\right\},-\tau^2\right]\right\}.$$

Here I_1 is the modified Bessel function of order one, ${}_2F_2$ is the generalized hypergeometric function, R_0 is the initial radius of the vortex ring and $\tau = R_0/\ell$ is the ratio of the outer radius to the diffusivity scale of the ring's core ℓ identified as the ring's inner radius. These invariants are found on the basis of the solution for the vorticity distribution

$$\omega_m = \exp\left(-\frac{1}{2}\left(\sigma^2+\eta^2+\tau^2\right)\right)I_1(\sigma\tau), \qquad (1)$$

$$\omega = \frac{\varsigma}{\varsigma_0},\ \ \sigma = \frac{r}{\ell}, \eta = \frac{x - x_o(t)}{\ell}, \tau = \frac{R_o}{\ell}.$$

Following the idea by Monseni and Gharib [2](see also Linden and Turner [3]), we use the dimensionless stroke length in the form

$$\frac{L}{D} = \sqrt{\frac{\pi}{2}}\frac{\Gamma^{3/2}I^{1/2}}{E}$$

This allows us to establish the dependence of the value of L/D on τ by equating the values of the impulse $I_m = M\rho$, circulation, and kinetic energy of the injected plug to the above-mentioned values of the viscous vortex ring. It is then possible to obtain the dimensionless energy and circulation for L/D=4

$$\tilde{E} = \frac{E_m}{\Gamma_m^{3/2} I_m^{1/2}} = 0.3,\quad \tilde{\Gamma} = \frac{\Gamma_m}{I_m^{1/3} U_m^{2/3}} = 2.1.$$

The evaluated values are in agreement with the experimental data. [1] However, unlike the Norbury vortices, the vorticity distribution (1) at short-time limit tends to the Gaussian form and is consistent with that observed experimentally.

THE 'FORMATION' NUMBER

The method of entrainment diagrams reported by Cantwell is applied. [5] For this purpose the equations for streamlines are determined by

$$\frac{dx}{dt} = \frac{1}{r}\frac{\partial \Psi}{\partial r} + U(t),\quad \frac{dr}{dt} = -\frac{1}{r}\frac{\partial \Psi}{\partial x},$$

The appropriate autonomous system in dimensionless variables σ, η, with two parameters $Re_0 = \Gamma_0 / \nu$ and τ, can be written as

$$\frac{d\eta}{ds} = -\frac{\eta}{2} + \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{Re_0\,\tau}{8} u_t,\quad \frac{d\sigma}{ds} = -\frac{\sigma}{2} + \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{Re_0\,\tau}{8} v_t. \qquad (2)$$

Here $s = \ln(t)$ and u_t, v_t are given by

$$u_t = \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^\infty \mu^2 F(\mu,\eta) J_1(\tau\mu) J_0(\sigma\mu)\, d\mu,\quad v_t = -\frac{1}{\sigma}\frac{\partial \Phi}{\partial \eta} = -\frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^\infty \mu\{-G(\mu,-\eta) + G(\mu,\eta)\} J_1(\tau\mu) J_1(\sigma\mu)\, d\mu,$$

where $F(\mu,\eta) = G(\mu,\eta) + G(\mu,-\eta), G(\mu,\eta) = \exp(\eta\mu)(1 - \mathrm{erf}(\frac{\mu+\eta}{\sqrt{2}}))$.

The analysis is focused on various isocline patterns of this dynamical system and on critical points (σ_0, η_0), where $d\eta / ds = 0, d\sigma / ds = 0$. The presented study is motivated by the hypothesis that analyzing of the system (2) can reveal properties of the nonlinear solution although the Stokes solution is used. Observing the particle trajectories for relatively high value of τ=10 versus Re_0, we recognize three regimes. In the first regime (Re_0< 150), the particle trajectories converge to a single node, which lies on the axis η. In the second regime (150<Re_0<700), particle trajectories are divided into two parts: some of trajectories converge to a node on the axis η, while other trajectories include the core of the ring. In the third regime (Re_0>700) all trajectories turn towards the center of the ring. The first transition can be interpreted as the transformation of the jet structure into the structure corresponding to the combination of a wake and a vortex ring. The appearing of these structures at the formation stage is consistent with the numerical studies. The solution (1) for high value of τ tends to the Gaussian distribution. When this solution is incorporated into the system (2) it shows the formation of the wake, i.e. indicates the nonlinear behaviour. We assume that the defined pattern with the second regime of the particle trajectories corresponds to the presence of a Gaussian distribution of vorticity inside the ring core. In this connection it is also natural to ask: 'What is the largest volume a vortex ring can achieve with the Gaussian distribution of vorticity in the ring core?' The boundaries for the flows consisting of the combination of a wake and a vortex ring in the parameter space (τ and Re_0) for $\tau \le 10$. are computed. This enables us to estimate the critical value of τ for which the above-mentioned structures disappear (for any Re_0). It lies in the range $3 < \tau < 4$. Since R_0 is fixed, the reduction of τ leads to the increasing of the ring inner radius and volume. Thus, the corresponding range of the limiting values of L/D for those the vortex ring has the largest volume can be estimated as $4.0 < L/D < 5.5$. Even though the above attempt includes heuristic arguments it gives the estimation of the 'formation' number. An estimation of this value, to our knowledge, not yet been suggested.

References

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