Two scale approach to anisotropic turbulence in HeI II

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Superfluid ^{4}He may be regarded as a mixture of normal fluid and superfluid, described by the velocity fields $\mathbf{V}_s$ and $\mathbf{V}_n$, and the density fields ρ_n and ρ_s, respectively. The rotation of superfluid is confined to one dimensional singularities called quantum vortices. Due to existence of these singularities, the superfluid component is coupled dissipatively to the normal one by the so-called mutual friction force $\mathbf{F}_{ns}$, which is proportional to the density of superfluid vortices L, and to the magnitude of counterflow $\mathbf{V}_{ns} = \mathbf{V}_n - \mathbf{V}_s$. At very low velocities the flows of normal and superfluid components are laminar and the quantum vortices form an ordered array of locally parallel lines; the whole system is described by the Hall Vinen Bakarevich Khalatnikov (HBVK) equations. At higher velocities the superfluid laminar flow develops into a superfluid turbulent flow in which quantum vortices form a chaotic tangle. The quasi isotropic tangle, which may be generated under the thermal counterflow, is described by the Vinen equation. There remain outstanding questions concerning highly anisotropic quantum turbulence, which may be produced mechanically (e.g. in Couette flow or spin-up), and which may not be described by either HBVK equations or Vinen equation. The theoretical difficulty in modeling anisotropic turbulence has been overcome recently be extensive numerical simulations in which the dynamics of quantized vortices is described by Biot-Savart law. Despite such simulation provide accurate description, they are restricted to relatively weak turbulence with low L.

In this study we propose an alternate approach: In macroscale scale we use the two-fluid model equations; the Euler equation for the superfluid component and the Navier Stokes equation for the normal component. These two equations are coupled by the mutual friction force $\mathbf{F}_{ns}$, which depends on the local line-length density L. The vortex density L at given point, say p, is calculated in a small volume Ω around p, based on the Vinen equation, which is modified to describe the evolution of line-length density of quantum tangle with net macroscopic vorticity. The generalized equation includes the drift of the anisotropic tangle caused by Magnus force. The volume Ω, must be enough large to contain a large number of vortex lines, but small when compared with the integral scale of the flow. It is assumed that the normal and superfluid velocity fields $\mathbf{V}_n$,$\mathbf{V}_s$, given in macroscale by Euler and Navier-Stokes equations, respectively, are constant in microscale, across small volume Ω. The proposed approach is, to some extent, similar to the non-local approximation (NLA), the numerical method proposed by Barenghi *et al.* (1997) to calculate the growth of the quantum tangle in imposed ABC flow of normal component. According to NLA the motion of vortex filaments is calculated using LIA, but in the superfluid velocity field, which is calculated in relatively small number of points using Biot-Savart law. In the incompressible approximation $\mathrm{div}\mathbf{V}_n = \mathrm{div}\mathbf{V}_s = 0$, the derived equations (after neglecting two terms in expressions for $\mathbf{F}_{ns}$ and $\mathbf{V}_L$) add up to the following closed system;

$$\rho_s(\frac{\partial \mathbf{V_s}}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s) = -\nabla p_s - \mathbf{F}_{ns} \ , \tag{1}$$

$$\rho_n(\frac{\partial \mathbf{V}_n}{\partial t} + \mathbf{V}_n \cdot \nabla \mathbf{V}_n) = -\nabla p_n + \mathbf{F}_{ns} + \eta \Delta^2 \mathbf{V_n} \ , \tag{2}$$

with mutual friction force $\mathbf{F}_{ns}$ and local line-length density L given by

$$\mathbf{F}_{ns} = \alpha\ \kappa\ \rho_s\ L\ \Big(\mathbf{q} \times (\mathbf{q} \times \mathbf{V}_{ns}) - \frac{2}{3} \mathbf{V}_{ns} (1 - q^2) \Big) \ , \tag{3}$$

$$\frac{\partial L}{\partial t} = \alpha\ I_0\ c_1\ (1-q^2)\ |\mathbf{V}_{ns}|\ L^{3/2} - \beta\ \alpha\ {c_2}^2\ (1-q^2)^2\ L^2 - \mathrm{div}\big(L\mathbf{V}_L\big) \ , \tag{4}$$

where $\mathbf{V}_L$ is the tangle drift velocity and $\mathbf{q}$, $|\mathbf{q}| = q \leq 1$, the vector of tangle polarization

$$\mathbf{V}_L = \mathbf{V}_s + \alpha\ \mathbf{q} \times \mathbf{V}_{ns} \ , \qquad \mathbf{q} = \frac{\nabla \times \mathbf{V_s}}{\kappa L} \, . \tag{5}$$

In the above α is the nondimensional coefficient of mutual friction, β depends logarithmically to the tangle density, but in most cases remains of order of quantized circulation $\kappa = 9.97 * 10^{-4}\text{cm}^2/\text{s}$, c_1, c_2, I_0 are geometric measures of quasi isotropic tangle (with $\mathbf{q} = 0$) adopted after Schwarz (1998), corresponding, respectively, to the average curvature of vortex lines, average curvature squared, and averaged anisotropy of the binormal. To derive expressions for mutual friction force $\mathbf{F}_{ns}$ and local line-length density L we follow the method proposed by Lipniacki (2001) and assume that the distribution of the unit tangent to vortex lines in the tangle is the most probable distribution which results in a given tangle polarization $\mathbf{q}$. In simple words, our assumption implies that macroscopic vorticity is absorbed into the tangle and causes its directional polarization, which is consistent with simulation of rotating turbulence using Biot-Savart law description by Tsubota *et al.* (2003). In the case of stationary rotating turbulence Eq. (4) gives the line-length density with respect to cylinder angular velocity $\Omega = \omega_s/2$ and counterflow velocity $\mathbf{V}_{ns}$ in implicit form

$$L = \frac{L_H}{(1-q^2)^2}\ , \qquad L_H = V_{ns}^2\Big(\frac{c_1\ I_0}{\beta\ c_2^2}\Big)^2, \tag{6}$$

where L_H is the steady-state line-length density resulting from same counterflow V_{ns} and $q = 0$. Anisotropy parameter q can be written as $q = L_\omega/L$, where $L_\omega = \omega_s/\kappa$ is the line-length density associated with rotation ω_s and zero counterflow. This prediction is the satisfactory agreement with Swanson *et al.* (1983) experimental data and simulations of Tsubota *et al.* (2003).

Equations (1-5) are supplemented by the boundary conditions for normal and superfluid velocity components, which in the simplest case, when there is no thermally induced flow, simplify to

$$\mathbf{n}\cdot\mathbf{V}_s|_{\partial U} = 0\ , \quad \mathbf{V}_n|_{\partial U} = 0. \tag{7}$$

Eq. (5) implies that when $\nabla\times\mathbf{V_s} \neq 0$, the vortex tangle (and the superfluid vorticity) moves across the counterflow. This implies that the boundary conditions must include additional condition connected with the rate of generation (or annihilation) of vortex lines at the wall. This condition must include some information about surface roughness. If the wall is perfectly smooth, the vortices can slip over and, provided that (7) is satisfied, there is no vorticity production at the wall; however, if the tangle drifts towards the wall, the line-length may annihilate. If the boundary is not perfectly smooth, pinning leads to the elongation of vortices, which results in a boundary production term proportional to the tangle density at the boundary.

The system of equations (1-5) was tested by applying it to the problem of formation of the shear flow between two parallel infinite material surfaces $z = 0$ and $z = D$. We consider the case, in which for $t < 0$ both material surfaces remain at rest, and then at $t = 0$, one of them starts moving along x axis with constant velocity V. We assumed that there is some initial density of remnant vortices L_0. The following scenario is observed. The normal component starts moving due to the viscosity forces, this introduces the velocity difference between normal and superfluids components $\mathbf{V}_{ns}$. The counterflow $\mathbf{V}_{ns}$ makes that the line-length density grows up (Eq. 4), and the two components become coupled by the mutual friction. The fact that superfluid velocity tends to match with normal velocity makes that $\boldsymbol{\omega}_s \neq 0$, which implies the polarization and drift of the tangle. After sufficiently long time the shear flow is formed.

There are three characteristic regimes with respect to the value of $D\ V$:

I - For large $D\ V$ the characteristic time of tangle generation is shorter than the drift time, in which the vortices can propagate across the vessel. This implies that the drift can be neglected. Moreover, until the components are not locked together, the tangle is almost anisotropic.

II - For small $D\ V$ the line generation is slow and the tangle is highly polarized. The drift term makes that vortex lines are efficiently swept out of the region where the counterflow exists. As the result there are two characteristic regions in the flow. In one of them the counterflow is almost zero and the quantum vortices form an ordered array of lines, and in the second region counterflow is non-zero, but there are few vortices.

III - For intermediate $D\ V$ the drift term in the modified Vinen equation is comparable with the generation term. There are three regions; of high tangle polarization in which the components are locked together, of intensive vortex generation, and of relatively dilute tangle, where the superfluid velocity is almost constant.

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