GENERIC HYDRODYNAMIC INSTABILITY

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Summary Every fluid dynamicist knows that almost all 3-d steady incompressible inviscid fluid flows are unstable; however, very few rigorous results about generic instability exist. Our idea is to use the geometry of the flow domain as a parameter. We prove that for generic geometry, *all* of the curl-eigenfield solutions to the steady Euler equations on R^3 (with periodic boundary conditions) are hydrodynamically unstable (linear instability, L^2 norm), with the possible exception of the zero-eigenvalue solution. The proof involves a marriage of topological methods with the instability criteria of Lifshitz-Hameiri and Friedlander-Vishik. An application of a new homology theory in symplectic geometry is the crucial step.

STATEMENT OF RESULTS

The problem of linear hydrodynamic instability for steady Euler flows on three-dimensional domains is classical in nature and foundational in implication [7, 9]. It is universally asserted that in dimension three such flows are almost always unstable, though the precise definition of *"almost always"* is an issue left undiscussed. The small literature on generic properties of fluid flows [6, 13] focuses on the Navier-Stokes setting and uses external forcing or Dirichlet data as a parameter.

We present a clear formulation of the problem and prove a generic instability theorem for a large class of flows — the curl eigenfields — which form the most fascinating and challenging steady solutions to the Euler equations. The chief difficulty with genericity issues for curl eigenfields is that the "space of all eigenfields" on a typical Riemannian three-manifold is a "discrete" space and is not amenable to perturbations. Our idea in formulating a well-defined genericity statement is to use the geometry of the domain as a parameter.

Theorem 1: *For a generic set of C^r Riemannian metrics on R^3 (for each $2 \leq r < \infty$) with periodic boundary conditions, all of the curl-eigenfield solutions to the Euler equations (with nonzero eigenvalue) are linearly hydrodynamically unstable in energy (L^2) norm.*

THE TOPOLOGY OF CURL EIGENFIELDS

The following technical result provides the basis for the instability theorem.

Theorem 2: *For generic choice of C^r metric (for each $2 \leq r < \infty$), all of the curl-eigenfields on a compact three-manifold M with non-zero eigenvalue have all fixed points nondegenerate. In addition, if any eigenfield does not possess any fixed points, then all its periodic orbits are nondegenerate.*

By nondegenerate fixed point, we mean that the linearization of the vector field about the fixed point has all eigenvalues nonzero. Nondegenerate periodic orbits are those whose Floquet multipliers are never equal to one.

The proof of Theorem 2 for fixed points is a combination of transversality theory with some machinery of K. Uhlenbeck [14]. The proof for periodic orbits is more delicate, using techniques from symplectic geometry and a theorem of C. Robinson [12].

Dealing with the case of curl-eigenfields without fixed points leads very naturally to a type of object in symplectic geometry called a *contact structure*. A contact structure on a 3-d domain is a plane field distribution which is maximally nonintegrable: it "twists" in all tangent directions.

The relationship between curl-eigenfields and contact structures is dual: the orthogonal plane field to any fixed-point-free curl eigenfield is a contact structure, and all contact structures arise as orthogonal plane fields of some curl eigenfield. The general version of this correspondence theorem can be used to understand the existence and qualitative behavior of steady solutions to the Euler equations [3, 4, 5]. For example, by exploiting the flexibility of contact structures, one can construct steady Euler flows on a compact 3-d domain which possesses periodic flowlines exhibiting all knot and link types [5].

OUTLINE OF PROOF

Thanks to an insight of Arnold and the analysis of Friedlander-Vishik [8] (who used the technique developed by Lifshitz-Hameiri [11]), it is now known that the underlying dynamics of the flowlines of the steady velocity field u can force linear instability. In particular, we rely on the following:

Instability Criterion: [8] *The presence of a nondegenerate periodic orbit of saddle-type in a steady Euler flow induces linear instability in the energy norm.*

By saddle-type, it is meant that there is are both expanding and contracting directions associated to the local flow about the orbit. Nondegenerate fixed points are always of saddle type in an incompressible flow; however, nondegenerate periodic orbits can be of *elliptic type*, with the flow rotating about the flowline. Such orbits foil the instability criterion.

The outline of the proof of the Theorem 1 is as follows: For a generic geometry, Theorem 2 guarantees that all fixed points present are hyperbolic. These are of saddle type and thus force instability. This, then, provides a quick proof of generic instability for invariant flow domains such as $S^2 \times [0,1]$ which are forced to have fixed points on the boundaries. The case of R^3 with periodic boundary conditions has vanishing Euler characteristic, and therefore admits flows without fixed points. The crux of the difficulty is determining when an eigenfield without fixed points possesses a saddle-type periodic orbit. Theorem 2 guarantees that all periodic flowlines are either of elliptic or saddle type: but how does one determine if periodic orbits exist and are of saddle type?

To answer this last, most difficult question, we turn to contact topological methods, including the recent *contact homology* of Eliashberg, Givental, and Hofer [2]. For an introduction to contact topological techniques in fluid dynamics, see [10]. Translated to the context of this application, contact homology is a method for counting periodic orbits of curl-eigenfields using topological properties of the orthogonal plane field. A delicate computation of the homology groups demonstrates that there must *always* exist a periodic orbit of saddle type in a curl eigenfield with nonzero eigenvalue.

Theorem 3: *Any nondegenerate nonvanishing curl-eigenfield on R^3 with periodic boundary conditions always possesses a closed flowline of saddle type.*

The proof of this is a very delicate computation using the methods of [1]. Combining these ingredients yields Theorem 1.

These techniques are an example of how topological methods can yield rigorous results about fluid flows without the need for restrictive hypotheses.

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