WEAK INERTIA AND MIXING BETWEEN ROUGH SURFACES

David Lo Jacono[1], Franck Plouraboué[2], Alain Bergeon[2]
[1] *Laboratory of Fluid Mechanics, Swiss Federal Institute of Technology Lausanne (EPFL), CH-1015 Lausanne, Switzerland.*
[2] *IMFT UMR 5502 / UPS, Allées du Pr. C. Soula 31400 Toulouse, France.*

INTRODUCTION

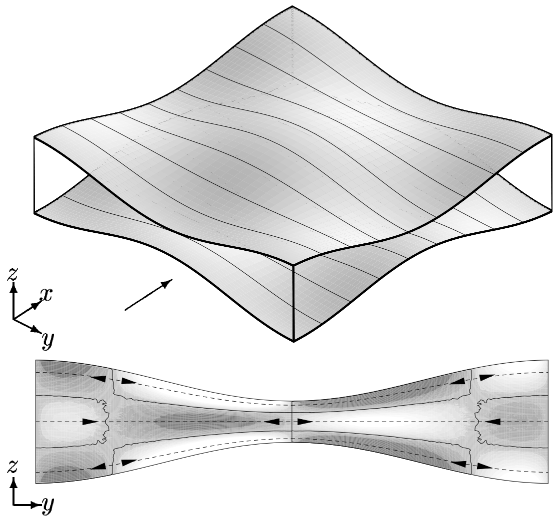

Figure 1. Top: tridimensional view of a patterned channel, the arrow indicates the mean flow direction. Periodic boundary conditions are used longitudinally whereas a Neumann impermeable flux is taken transversely. Arrow indicates the passive scalar displacement. Bottom: Lagrangian displacement map in a Poincaré section (y, z) of the flow.

Weak inertial effects in flows between two rough surfaces may lead to significant effects in various situations such as fracture flow or micro-fluidic. In the latter, Reynolds numbers are generally small and the flow is laminar. Recent experimental results [1] have shown that proper surface patterning could produce vortex which are transverse to the longitudinal mean flow and could be used to produce chaotic stirring [2]. Results of these experiments have been analyzed successfully in the context of Stokes flows in the limit of small patterning amplitude [3]. The question of finding optimal patterning for mixing remains an important issue for many practical applications. It requires theoretical investigations to avoid expensive computations of three dimensional Navier-Stokes equations in complex geometries. Hence, another interesting limit for surface patterning is the case of smooth variations of the aperture field between the two solid surfaces. If the Reynolds number is zero, the lubrication approximation is valid. The flow has a zero helicity and transverse mixing does not occur. Nevertheless including inertia could lead to efficient transverse mixing. Different works have focused on the impact of inertia on mixing issues in some related context [4, 5]. The present study concerns the influence of weak-inertia effects produced by a smooth surface patterning on the flow field and the streamline geometries. The analysis presented here is a first step towards optimization of the surface patterning associated with inertially driven stirring. First section describes an asymptotic treatment of the Navier-Stokes equation that leads to inertial corrections to lubrication equations. Second section gives some example of inertial influence on the flow field properties that are relevant for mixing.

ASYMPTOTIC ANALYSIS OF INERTIAL EFFECTS

We consider two surfaces with relative vertical distance $2Z_-(x, y)$ where (x, y) are the horizontal Cartesian coordinates. An example of such surface patterning is depicted in figure 1. We assume that the surfaces are smooth, so that the local slopes of each surfaces are of order $\epsilon \ll 1$. The Reynolds number Re is considered to be larger than one, whereas the "lubricated" Reynolds number $Re\epsilon$ is considered to be small. Expanding the pressure p (non dimensionalyzed by the viscous pressure) and the velocity field $\mathbf{U} = (u, v, \epsilon w)$ (non dimensionalized by the mean velocity) in powers of ϵ in the Navier-Stokes equations leads to the following development:

$$\begin{array}{lllllll} p &=& \epsilon^{-1} p_0 &+& Re\, p_1 &+& O(\epsilon^2, Re^2\epsilon), \\ \mathbf{U} &=& \mathbf{U}_0 &+& Re\epsilon\, \mathbf{U}_1 &+& O(\epsilon^2, Re^2\epsilon^2). \end{array} \tag{1}$$

The zeroth order gives the lubrication approximation with uniform pressure $p_0(x, y)$ along the vertical direction for which the mean flux $\mathbf{q_0}$ – the integrated horizontal in-plane velocity $\mathbf{u_0} = (u_0, v_0)$ in between the surfaces – is proportional to the pressure gradient: $\mathbf{q_0} = -2\nabla p_0 Z_-^3/3$. The leading order lubrication velocity field $\mathbf{u_0} = -1/2\nabla p_0(z^2 - Z_-^2)$ is also proportional to the pressure gradient so that the helicity $H_0 = \mathbf{U_0} \cdot \nabla \times \mathbf{U_0}$ of the associated flow field is zero ($H_0 = 0$), while self-consistently neglecting $O(\epsilon)$. The inertial correction gives a non-zero $O(Re)$ contribution to the helicity, so that transverse mixing perpendicular to the mean flow becomes possible. Inserting (1) in the Navier Stokes equations leads to the following in-plane correction to the flow field:

$$\mathbf{u_1} = -(z^2 - Z_-^2)\left(\frac{1}{2}\nabla p_1 + Re\varepsilon(z^4 - 4z^2 Z_-^2 + 11 Z_-^4)\left(\frac{1}{240}\nabla(\nabla p_0)^2 + \frac{1}{Z_- 60}\nabla p_0(\nabla p_0 \cdot \nabla Z_-)\right)\right), \tag{2}$$

the integration of which, associated with the divergence-free flux condition, gives the two-dimensional equation for the pressure correction $p_1(x,y)$:

$$\nabla \cdot \left(Z_-^3 \nabla p_1\right) = -\frac{6}{5} Re\epsilon \nabla \cdot \left(\nabla(\nabla p_0)^2 \frac{Z_-^7}{63} + 4\nabla p_0 (\nabla p_0 \cdot \nabla Z_-) \frac{Z_-^6}{3} \right) \tag{3}$$

We call equation (3) Oseen-Poiseuille equations as it gives the first in-plane inertial correction to the lubrication Darcy-Poiseuille flux. Beside two-dimensional, inertial effects are not simply related to the lubrication pressure field $p_0(x,y)$ and the aperture field $2Z_-(x,y)$. The high polynomial order dependence associated with the right hand side of equation (3) requires the use of a high order numerical method to solve it.

MIXING ISSUE

We use a spectral element method to solved the simplified boundary layer equations introduced in the preceding section. Figure 2 shows the computed stream-lines mean flux $\mathbf{q}$ of the velocity field in the horizontal plane. When changing the sign of the mean applied pressure difference, the zero Reynolds number flow shows no difference due to the Stokes flow reversibility.

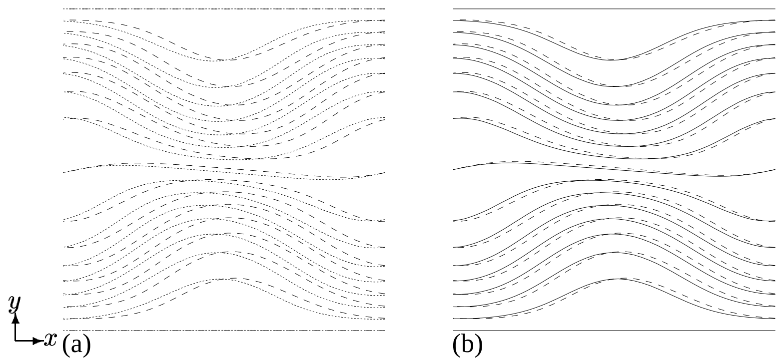

Figure 2. Mean in-plane flux $\mathbf{q}$ stream-lines are displayed. ——: $\mathbf{q_0}$, - - -: $\mathbf{q_0} + (\epsilon \mathbf{Re})\mathbf{q_1}$. The direction of the mean applied pressure gradient has been switched from left to right in (a), to right to left in (b)

On the contrary, when inertia corrections are taken into account, the stream-lines structure is no more reversible. Some symmetries of the non-inertial stream-lines geometry are broken and the two inertial stream-lines associated with opposite pressure gradient are symmetric with respect to the broken symmetry. Another interesting feature of the flow field appears when examining Poincaré sections in the (y,z) plane as represented in figure 1. When $Re = 0$, there is no transverse mixing. When weak inertia is added, the vertical dependence of the flow field is no longer spatially uniform leading to a Lagrangian transverse displacement presented in figure with a grey scale. The continuous lines drawned onto the grey scales are the zero-displacement level sets in the transverse direction. The arrow indicates the *stability* associated with fixed points with zero-displacement. The rich set of attracting and repulsing regions observed in this figure illustrates that a proper succession of surfaces patterning induces a succession of stretching/folding which are necessary for Lagrangian chaos.

References

[1] Stroock A.D., Dertinger S.K.W., Ajdari A., Mezic L., Stone H.A., Whitesides G.M.: Chaotic mixer for microchannels. *Sciences* **295**(5555):647–651, 2002.

[2] Aref H.: Stirring by chaotic advection. *J. Fluid Mech.* **143**:1–21, 1984.

[3] Stroock A.D., Dertinger S.K., Whitesides G.M., Ajdari A.: Patterning flows using grooved surfaces. *Anal. Chem.*, **74**(20):5306–5312, 2002.

[4] Jones S.W., Thomas O.M., Aref H.: Chaotic advection by laminar-flow in a twisted pipe. *J. Fluid Mech.* **209**:335–357, 1989.

[5] Sobey I.J.: Dispersion caused by separation during oscillatory flow through a furrowed channel. *Chem. Eng. Science*, **40**(11):2129–2134, 1985.