MICROSTRUCTURE OF A DILUTE SEDIMENTING SUSPENSION

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Summary We consider the sedimentation of a dilute suspension of particles in a viscous incompressible fluid at low Reynolds number. Starting from a general Liouville type evolution equation for the particle configuration probability density and assuming finite range of correlations, we are able to explicitly find the two-particle stationary distribution function for a sedimenting suspension in the low particle concentration limit.

INTRODUCTION

Sedimentation is the movement of a suspension under a given external force field acting on the mesoscopic particles suspended in a fluid. The slow sedimentation of a monodisperse, dilute suspension of non-Brownian rigid particles in a viscous fluid where inertial effects can be neglected is one of the basic and fundamental problems in microhydrodynamics of suspensions. However simple it might seem at first glance, it features a myriad of interesting phenomena and questions which are still unresolved and need answering.

A single sphere sedimenting in a viscous fluid falls with the well known constant Stokes velocity. Due to the long range and multi-particle hydrodynamic interactions between an ensemble of sedimenting particles the mean suspension velocity decreases rapidly with increasing particle volume fraction ϕ. The first order correction -6.55ϕ to the single particle sedimenting velocity was first computed incorporating a renormalization procedure by G. K. Batchelor [2]. The calculation assumed: (i) a low particle Reynolds number justifying the neglect of inertia, (ii) consideration of only two-body interactions, (iii) a random (or equilibrium) distribution of particles in configuration space, and (iv) the system infinite in the direction transverse to the force field. But as R .E. Caflisch and J. H. C. Luke [4] pointed out, a random distribution of sedimenting spheres results in velocity fluctuations (variance) which scale linearly with the macroscopic size of the system. The above theoretical predictions have not been seen in experiments. In contradictions [5], [6], [7] [8] experimental measurements of the velocity fluctuations have consistently been interpreted as having a saturation level at some critical and universal system size proportional to the mean interparticle spacing, leading to fluctuation scaling independent of system size. On the other hand, virtually all numerical studies reported [9] have been consistent with the theoretical scaling proportional to the system size.

This obvious discrepancy has attracted a lot of theoretical attention. The first to propose a solution were D. L. Koch and E. S. G. Shaqfeh [10]. Their central idea based on the violation of the assumption (iii). Due to three-particle interactions, the particle were to rearrange in such a way, as to cancel out, or screen, the long rang hydrodynamic interactions. An explicit form of the particle distribution function exhibiting the above property was calculated, but the predicted correlation length scaling turned out not to agree with the experiment.

Several other attempts have been made to crack the mysterious inconsistency. Brenner [11] emphasized the importance of side wall effects on the microscopic evolution, and possible screening of hydrodynamic interactions. Further calculations [12], [13], [14] incorporate the idea of a vertical stratification owing to the broadening of the sediment front. Non of these seem to be experimentally verifiable nor to present a consistent and general picture.

Recent experiments [15] indicate that the particle distribution while sedimenting is not the random or equilibrium one. The explicit determination of the statistical distribution of a steady state sedimenting suspension is still an open problem, with which we are to deal in this work.

METHOD AND MAIN RESULTS

We consider a suspension of N particles sedimenting in a viscous fluid of viscosity η. The radius a of the particles is of the order $10 - 10^2 \mu m$ insuring that the Peclet number is large and that Brownian motion can be neglected.

The basic equation governing the evolution of a suspension is the Navier-Stokes equation, which under the above assumptions can be reduced to the linear Stokes equation. It is further assumed that a given force field $F = (\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_N)$ acts on the particles. Since the problem is linear, the relation between the individual particle velocities and the forces is also linear. This relation can be summarized in a $3N \times 3N$ mobility matrix $\mu(X)$ dependent on the given configuration $X = (\boldsymbol{R}_1, \ldots \boldsymbol{R}_N)$ of particles.

Looking at the problem from a statistical physicists point of view, it is useful to consider the statistical distribution $P(X,t)$ for finding the system in a given configuration X at time t. This distribution satisfies the Liouville equation:

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial X} \cdot (\mu \cdot FP) = 0. \qquad (1)$$

Introducing next the cluster expansion of the general mobility matrix we are able to derive a BBGKY (Bogolubov-Born-Green-Kirkwood-Yvon)-type hierarchy of equations [1] governing the time evolution of reduced s-particle distribution functions $n_s(\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_s; t),\ s = 1, \ldots, N$, that is functions representing the probability of finding pairs, triplets, etc. of particles in given configurations. They are defined by

$$n_s(\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_s; t) = \frac{N!}{(N-s)!} \int d\boldsymbol{R}_{s+1} \ldots d\boldsymbol{R}_N P(X;t), \qquad s = 1, 2, \ldots, N. \tag{2}$$

The hierarchy equations give a relation between the time derivative of the s-particle partial distribution function and the spatial derivatives of the mobility matrix and all the consecutive $s, s+1, \ldots$-particle partial distribution functions and their spatial integrals in respect to particle positions with labels $s+1, \ldots$.

In further considerations we focus on stationary states. It can be explicitly shown that the equilibrium state, or random configuration probability density is not a solution of equation (1). Moreover, assuming a random distribution we encounter, discussing the hierarchy equation, similar divergence problems as R .E. Caflisch and J. H. C. Luke [4] came upon while calculating the velocity variance. In this case, due to the very long range character of the hydrodynamic interactions some terms in the hierarchy become infinite in the thermodynamic limit (i.e. the limit of the system volume infinite, number of particles infinite while keeping the particle concentration constant).

The main idea behind this work is to find a particle distribution which despite long range hydrodynamic interactions, will in a self-consistent way keep the correlations length finite (i.e. the correlations will decay faster than $1/r^3$, where r is the interparticle distance) and thus cancel out the otherwise infinite terms in the equations.

We concentrate on the limit of small particle concentrations, when only terms in the lowest order in particle density are taken into consideration and the infinite hierarchy of equations describing the time evolution of the reduced distribution functions becomes a set of closed equations. The first nontrivial equation is the one giving the time evolution of the three-particle reduced distribution function $n_3(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)$. The requirement to retain a finite correlation length, in spite of long range terms in the interactions between particles, together with the explicit formula for the three-particle mobility matrix given in terms of Green operators [3] lead to a simple linear partial differential equation for the two-particle reduced distribution function. The received equation is solved, and the explicit form of the two-particle distribution function given. Results and discussion will be presented

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