DRIFTING AND MERGING COLLARS IN LIQUID-LINED TUBES

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Summary Lubrication theory is used to describe the evolution under surface tension of an axisymmetric air-liquid interface in a long, liquid-lined tube. The resulting structures and their interactions are investigated, leading to a new similarity solution to the thin-film equation and an understanding of the mechanism of collar drift.

INTRODUCTION AND MODEL

The evolution of a thin, viscous liquid lining that coats the interior of a cylindrical tube has many applications including modelling the liquid lining of a lung airway, the oil/water interface in a sandstone pore, industrial painting and other coating flows. We consider a cylindrical tube of radius a lined on its interior with a thin liquid film of thickness $a\epsilon h(z,t)$, where $\epsilon \ll 1$, az measures the distance along the tube and $(a\mu/\gamma\epsilon^3)t$ measures time. The film evolves under surface tension γ and viscosity μ. Gravitational effects are neglected, and the core fluid in the tube is assumed to be dynamically passive. Following Hammond [1], lubrication theory yields the following evolution equation for h:

$$h_t + \tfrac{1}{3}(h^3(h_z + h_{zzz}))_z = 0. \tag{1}$$

We solve this numerically in $0 \le z \le L$ subject to no-flux and symmetry conditions $h_z = h_{zzz} = 0$ at $z = 0$ and $z = L$. Equation (1) models both the Rayleigh instability of the liquid lining of a tube and the Rayleigh-Taylor instability of a thin film on the underside of a plane.

NUMERICAL RESULTS

Characteristically, solutions to (1) are unstable to long-wavelength perturbations and form growing liquid 'collars' of baselength 2π separated by smaller draining 'lobes'. Figure 1a) illustrates the formation of the collars and lobes at early times ($t \le 50$, $L = 6\pi$), and Figure 1b) shows how the central fluid collar subsequently grows and moves, drifting firstly to the right and then to the left. By tracking the heights and axial locations of the extrema of film thickness, we find that

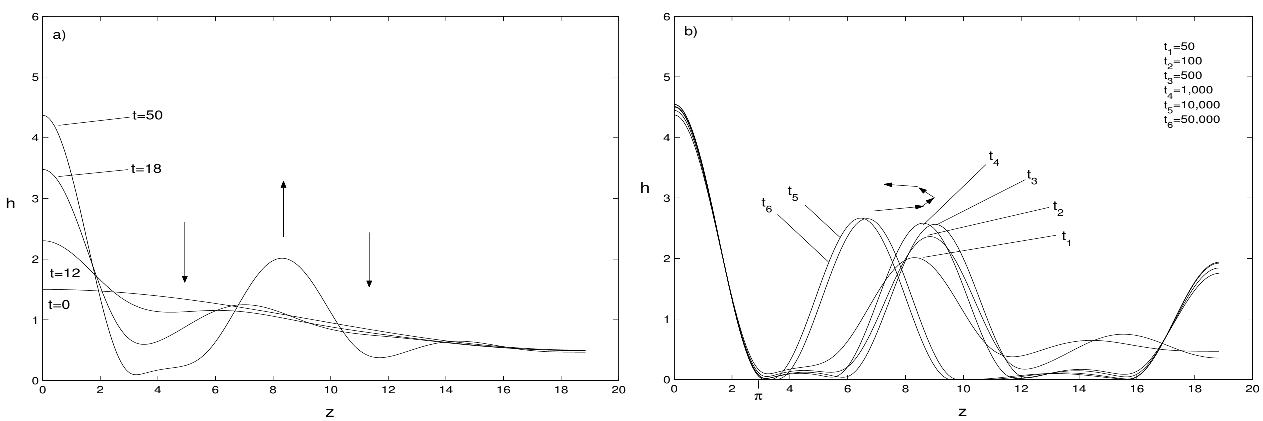

Figure 1. Solutions to (1) subject to no-flux conditions. a) and b) show the height profiles at early and late times respectively.

the height in the shrinking draining lobe situated near $z = \pi$ evolves (at large times) at a rate contrary to the scalings suggested by Hammond [1]. Motivated by our numerics we seek a similarity solution to describe this structure.

ASYMPTOTIC ANALYSIS

In our region of interest $|z - \pi| \ll 1$, as $h \ll h_{zz}$ the evolution equation (1) becomes the well known 'thin-film equation'

$$h_t + (h^3 h_{zzz})_z = 0. \tag{2}$$

Scalings derived from our numerics suggest a change of variables, and we define $\xi = (z-\pi)t^{\frac{1}{2}}$ and $f(\xi) = ht$. Examining plots of ht for various time levels from our PDE data (see figure 2a), we observe that a self-similar structure is present. Substituting $f(\xi)$ into the thin-film equation (2) yields

$$-f + \frac{\xi}{2}f' + (f^3 f''')' = 0. \tag{3}$$

Matching to the adjacent quasi-static collars yields the boundary conditions $f \sim \kappa_1\xi^2/2 + O(1)$ as $\xi \to -\infty$ and $f \sim \kappa_2(\xi - \xi_0)^2/2 + O(1)$ as $\xi \to \infty$. κ_1 and κ_2 are the given curvatures of the left and right collars respectively and ξ_0 is the location of the effective contact line of the right-hand collar. The numerical solution to the time-dependent problem at $t = 10,000$ is used as an initial profile to solve for f. The resulting similarity solution to the thin-film equation (plotted alongside the initial profile) is shown in figure 2b). The numerical solution was checked and found to be insensitive both

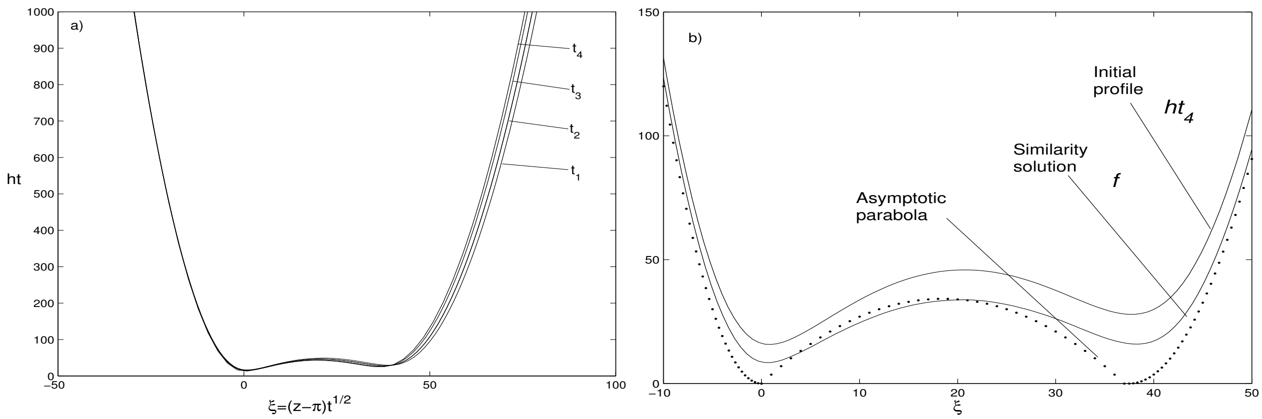

Figure 2. a) Illustration of self-similar behaviour at times $t_{1,2,3,4} = 1000, 2000, 5000$ and 10000, and b) novel similarity-solution alongside the chosen initial profile and asymptotic prediction for the internal parabola.

to grid size and the length of the spatial domain used. Thus we conclude that the similarity solution is a genuine physical structure and we conjecture that the draining lobe that separates the two larger growing collars persists as $t \to \infty$. The presence of this lobe creates a dissipative barrier that prevents the free flow of fluid from one collar to the other, thus preventing the collars from merging.

To understand (3) in greater detail, we examine both a symmetrical, one-parameter family of solutions and an asymmetric, two-parameter family of solutions (not shift invariant) to (3). We find that there are three standard solution profiles depending on the region of parameter-space within which the solution lies. They are a) parabolic or U-shaped solutions, b) W-shaped solutions, and c) solutions which exhibit 'touchdown' behaviour ($h \to 0$ as $\xi \to \xi_t$ for some ξ_t). It is the W-shaped solutions that we are most interested in as these pertain to a draining fluid lobe that separates two larger, growing collars (Figure 2a). We find in some cases that numerical W-shaped solutions can be approximated asymptotically, assuming quasi-steady behaviour near the local minima; we can recover for example the approximate form of the draining lobe (see the dotted line in Figure 2b) as a function of κ_1, κ_2 and ξ_0.

We also consider the drifting of a single fluid collar, using an approach similar to that used by Glasner and Witelski [2]. We write the fluid height as $h = \bar{h} + \hat{h}$, the equilibrium collar solution $\bar{h}$ plus a small perturbation $\hat{h}$, and then substitute into the evolution equation (1). An adjoint method yields a solvability condition. Two solutions of this problem are sought, one even and one odd. The even solution gives a mass conservation law, whereas the odd solution may be shown to yield an equation that implies the direction of drift of a fluid collar is dependent on the extremal heights at the leading and trailing edges of the collar, with the collar drifting in the direction of the collar's greater minimal height.

CONCLUSIONS

A new group of similarity solutions to the thin film equation has been found. This suggests that a draining lobe that separates two growing collars can act as a barrier, prohibiting any potential merging of the collars. The direction of drift of a single fluid collar is dependent on the minimal heights at the collar's leading and trailing edges.

References

[1] Hammond P.S.: Nonlinear adjustment of a thin annular film of viscous fluid surrounding a thread of another within a circular pipe. *J. Fluid Mech* **137**:363–384, 1983.

[2] Glasner K.B., Witelski T.P.: Coarsening dynamics of dewetting films. *Physical review E* **67**:(016302)1-12, 2003.