

ON THE NATURE OF VIRIAL LIAPUNOV FUNCTIONAL IN HYDRODYNAMIC INSTABILITY.

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1. THE EXTENDED SUMMARY.

A dream of generations of researchers is to obtain upper and/or low bounds of growing solutions in both classical and continuum mechanics without actual solving the governing equations. One problem, where such approach is used is the ‘magnetic dynamo’. Obtaining of various low bounds has been moved to a center of interest in fluid dynamics since the problem of the finite time singularity for the solutions of Euler or Navier Stokes equations has attracted so much attention. Related approach in the stability theory is known as the ‘direct Liapunov method’. For the cases of instability it is based on the use of quantities called ‘Liapunov functions’ or ‘Liapunov functionals’, which grow with time and provide low bounds for growing perturbations. First efficient ‘Liapunov function’ was introduced by Liapunov himself to proof instability of an equilibrium solution of a finite-dimensional Hamiltonian system. The ‘Liapunov function’ in those cases has been defined as scalar product between perturbations of generalized momentum $p(t) = (p_1(t), \dots, p_n(t))$ and coordinates $q(t) = (q_1(t), \dots, q_n(t))$:

$$(1.1) \quad W \equiv pq$$

In fluid dynamics ‘Liapunov functionals’ were introduced and used for studies of instability of equilibria of general fluid and ‘fluid+solid’ systems, as well as for plasma instabilities. The ‘Liapunov functional’ in those cases represents a natural heuristic infinite-dimensional generalization of the original Liapunov’s form (1.1). For example for an inviscid homogeneous fluid it is defined as scalar product between perturbations of velocities $\mathbf{u}(\mathbf{x}, t)$ and Lagrangian displacements $\xi(\mathbf{x}, t)$ of material particles,

$$(1.2) \quad W \equiv \int_{\tau} \mathbf{u} \xi d\tau$$

where integral is taken over the whole volume of fluid.

Many results for fluids had been obtained for linearized conservative systems. In other papers this type of Liapunov functional have been generalised and applied to exact nonlinear problems, to viscous fluids and to some ‘complex’ fluids. The use of the direct Liapunov method has been always restricted by the cases of instability of equilibria or effective equilibria. At the same time expressions similar to (1.1), (1.2) have been in active use in molecular physics and in astrophysics for long time. In those areas they are often called ‘virials’, and are in use for deriving the equations of state for molecular or star gases. One should notice that such a use is right opposite to targets of our paper: at a thermodynamical equilibrium the average value of (1.1) is accepted to be zero, while we are interested only in situations when it is growing. Despite of that difference in our paper we use the same terminology: namely (1.1) is

called ‘virial function’ and all its heuristic generalizations to continuum mechanics (e.g. (1.2)) called ‘virial functionals’. Historically ‘virial equalities’ were invented in classical mechanics in so called ‘virial theorem’ and in the ‘Clausius virial’.

It is remarkable, that all quoted above papers do not contain any attempts to explain the nature of the used virial functions or functionals. It has been the main reason for spreading of a persistent myth of ‘irrationality’ and of a ‘mysterious nature’ the Liapunov functions or functionals for the cases of instability. Such unusual situation represents a sharp contrast to the proofs of stability, where existing integrals of motion (energy, momentum, vorticity, etc.) provide a broad basis for constructing of Liapunov functionals and obtaining of various upper bounds for solutions.

The aim of the present paper is to present a systematic view on the nature of the ‘virial type’ Liapunov functionals for various areas of fluid mechanics. We recover close link between virial equalities and the well-known in differential geometry ‘Jacobi equation for geodesic deviations’ [1]. Namely ‘virial’ represents a ‘distance’ between two geodesics which grows for the cases of instability.

Following to our approach we are able to introduce expressions for ‘virial’ and ‘virial equality’ (and to analyze if they are useful) for a fluid system as soon as we know expression for its Lagrangian or Hamiltonian. Technically, our approach is based on the use of standard for the calculus of variations one-parameter families of vector fields, depending on a scalar parameter ϵ . Variations of velocities and other fields correspond to derivatives with respect to ϵ . Our ‘virial equalities’ appear as equalities following from the equations for the second derivatives of Lagrangian with respect to ϵ , or, in other words, from the equations for the second variation of the action.

Our ‘virial equalities’ are valid by virtue of the exact equations of motion and represent general relations between fields which can be used to investigate not only stability, but also general properties of fluid flows. We use ‘virial equalities’ in order to obtain *a priori* estimates (lower bounds) of solutions and instabilities. In the linear approximation, when a state of rest is taken to be the basic solution, our ‘virial equalities’ coincide with the conventional ones. Our approach recovers and unifies all known instability results obtained using ‘virials’ and additionally shows that classical instabilities of steady fluid flows such as the ‘horseshoe instability’, as well as instability considered earlier by Arnold, can be recovered using our ‘virial equality’. Covering of those classes of flow instabilities by the ‘virial method’ represent a qualitatively new result, since of all previous related results were obtained for instabilities of various equilibria (states of rest).

In this paper, we shall derive equalities for time derivative of ‘virials’ for a number of dynamical systems of fluid mechanics and employ them to obtain various sufficient conditions for instability. The outline of the paper is as follows. First, we consider similar approach in classical mechanics. Second, we introduce a virial functional for general steady or unsteady flows of an inviscid incompressible fluid in a fixed domain and use this functional to obtain sufficient conditions for instability. Then we extend the results to free-surface flows, stratified flows (in Boussinesq approximation) and magnetohydrodynamic flows. Finally we present discussion and conclusions. A part of presented results has been published in [2].

REFERENCES

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