STABILITY OF STRATIFIED SHEAR FLOWS WITH A MONOTONIC VELOCITY PROFILE WITHOUT INFLECTION POINTS

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Summary In the two-layer model of a stably stratified medium the stability of flows without inflection points on the monotonic velocity profile is considered.

In a stratified medium, the Rayleigh and Fjørtoft theorems that play a very important role in stability theory of homogeneous flows, are invalid. There is a well-known example: stratified flow with a velocity profile $V_x = U_0 \sinh y$ is unstable [1]. This paper presents a number of results pertaining to the stability of a wide class of flows without inflection points on the velocity profile in a stably stratified medium.

Let us consider the flow above a solid bottom $y = 0$ in the gravity field g. An ideal two-layer fluid has the density

$$\rho(y) = \begin{cases} \rho_1, & 0 \le y < y_N, \\ \rho_2 < \rho_1, & y_N < y < \infty, \end{cases} \qquad N^2(y) = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}y} = J\delta(y - y_N),$$

and a monotonic continuous velocity profile without inflection points,

$$V_x = u(y), \quad u''(y) \le 0; \quad u(0) = 0, \quad u'(0) = 1, \quad u \to 1 \ \text{as} \ y \to \infty,$$

where $N^2(y)$ is the squared Brunt–Väisälä frequency, and the prime denotes the derivative with respect to y. Within the Boussinesq approximation, the stream function $\psi = f(y)\exp(-i\omega + ikx)$ satisfies the Rayleigh equation, with the matching condition when $y = y_N$, and the boundary conditions:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - \left(\frac{u''}{u - c} + k^2\right) f = 0; \qquad J = (c - u_N)^2 \left(\frac{f'_-}{f_-} - \frac{f'_+}{f_+}\right)_{y=y_N}; \quad f_-(0) = 0; \quad |f_+| < \infty; \tag{1}$$

where $f_\pm$ stands for the solutions of the Rayleigh equation to the right and left of y_N, $c = \omega/k$, $u_N = u(y_N)$. When $k = 0$, the problem (1) has a continuous spectrum of neutral oscillations

$$J \int_0^{y_N} \frac{\mathrm{d}y}{(u - c)^2} = 1, \qquad u_N < c < \infty. \tag{2}$$

Oscillations with $c \ge 1$ remain neutral for any k and obey the dispersion equation $J = J_c(k)$. Oscillations with $c < 1$, however, become unstable when $k > 0$ (except for the NCP-modes, see below). It can be shown that at any $k > 0$ the $c = 1$ mode is marginally stable and there are no other marginally stable modes but for "ultimate" one, with $J = 0$ and $c = u_N$ for any k.

Consequently, flow becomes unstable with an arbitrarily small density difference, and this occurs at any k, and it is unstable in the band $0 < J < J_1(k)$ (see the figure). A numerical calculation of stability of the flows $u = \tanh y$ and $u = 1 - \mathrm{e}^{-y}$ confirmed this conclusion.

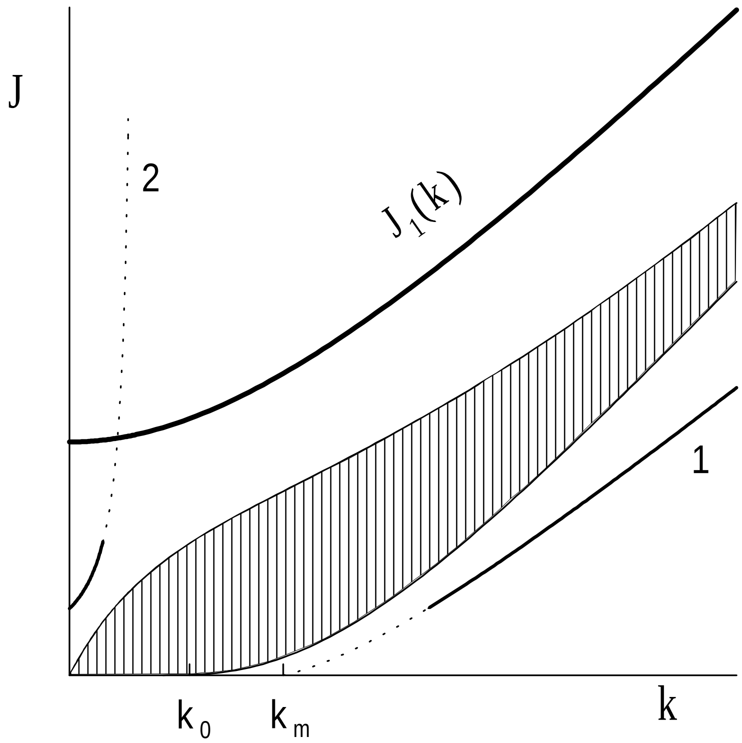
If on the velocity profile there are null-curvature points (NCP) $y = y_m$, at which u'' has zero (of an even order), this picture should be supplemented with several substantive details. It is easy to show that:

i) the neutral oscillation of a homogeneous ($J = 0$) medium with the phase velocity $c = c_m \equiv u(y_m)$ corresponds to each NCP, and its wave number k_m is the left-hand boundary of the branch $J = J_+^{(m)}(k)$ of neutral oscillations having the phase velocity c_m (curve 1 in the figure);

ii) if $c_m > u_N$, there is another (long-wavelength) branch of neutral oscillations, $J = J_-^{(m)}(k)$ (curve 2 in the figure) satisfying when $k = 0$ equation (2) with $c = c_m$; if $c_m < u_N$, in some cases this long-wavelength branch does also exist. Let us name these branches of neutral oscillations the NCP-modes.

In calculating the corrections for δk and δJ to the phase velocity c of the NCP-mode, we see that if $c_m < u_N$, there are no other solutions to (1) in its neighborhood. If, however, $c_m > u_N$, the segments of curves 1 and 2, shown in the figure by solid lines, are bordered (on both sides!) by unstable solutions of (1), while in the neighborhood of the segments, shown by dashed lines, there are no solutions to (1).

In other words, the NCP-modes in the case $c_m < u_N$ and/or the NCP-oscillations shown by dashed lines (in the case $c_m > u_N$) complement the previously determined spectrum of the problem (1): along with them, inside the instability band $0 < J < J_1(k)$ for the same J and k there necessarily is an unstable oscillation, and above the band there is a neutral oscillation, with $c \ge 1$. Conversely, the oscillations (with $c = c_m > u_N$), shown by solid lines, belong to the previously determined continuous spectrum. On the solid lines the growth rate



becomes zero, so that they look as if they "cut" the instability region. Numerical calculations provide support for this.
In the often used model flow,

$$u(y) = \begin{cases} y, & 0 \le y < 1, \\ 1, & 1 < y < \infty, \end{cases}$$

the entire velocity profile consists of NCP, and the NCP-modes produce a dense set, substantially narrowing the instability region (appears shaded in the figure). When $J \to +0$, only the bounded spectrum of disturbances is unstable, $0 < k < k_0$ ($k_0 = k_m$ when $c_m = u_N$), while the upper and lower boundaries of the instability region are envelopes of the families of curves $J = {J_-}^{(m)}(k)$ and $J = {J_+}^{(m)}(k)$, respectively.

References

[1] Thorpe S.A.: Neutral eigensolutions of the stability equation for stratified shear flow. *J. Fluid Mech.* **36**: 673 – 683, 1969.