TRANSVERSE FLOWS IN RAPIDLY OSCILLATING CYLINDRICAL TUBES

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Summary Motivated by the problem of flow in collapsible tubes, we analyse the flows that develop in the cross sections of fluid-conveying pipes whose walls perform high-frequency oscillations. Using numerical and asymptotic methods, we show that the velocity perturbations induced by the wall oscillation are dominated by their transverse components. The transverse velocity field consists of an inviscid core flow and thin Stokes layers near the wall. The total viscous dissipation is shown to depend sensitively on the mode shapes of the wall oscillation.

INTRODUCTION

Many physiological flows (e.g. blood flow in the veins and arteries) are strongly affected by the interaction between the fluid flow and the vessel wall elasticity. The problem of flow in collapsible tubes has therefore received much interest in the biofluids research community (see, e.g., [1] for a recent review). Experimentally, the problem is typically investigated with a 'Starling resistor', a device in which fluid is driven through a finite-length, thin-walled, elastic tube which is mounted on two rigid tubes and enclosed in a pressure chamber. One of the most striking features of this system is its propensity to develop large-amplitude self-excited oscillations. Due to the complexity of the system (an unsteady finite Reynolds number flow, interacting with the large displacements of a non-axisymmetrically buckling cylindrical shell), our understanding of the mechanism(s) that initiate and maintain these oscillations is still limited.

In a study of a 2D model problem (high Reynolds number flow through a channel in which part of one wall is replaced by a highly pre-stressed elastic membrane), Jensen & Heil [2] have recently provided a rational asymptotic description of an instability that causes the development of high-frequency oscillations. The asymptotic predictions were confirmed by direct numerical simulations which showed that the mechanism that is responsible for the initial instability also controls the large-amplitude oscillations that develop subsequently. At leading order, the flow consists of an inviscid core flow, which represents the axial 'sloshing' of the fluid that is displaced by the transversely oscillating membrane. Thin Stokes layers form on the channel walls. A key ingredient for the instability mechanism is the fact that the inviscid core flow can create a net influx of (kinetic) energy into the system. The development of an instability depends crucially on the ratio of this influx of energy to the viscous dissipation in the Stokes layers.

The main aim of the present study is to investigate how (and if) the 2D instability mechanism can be adjusted to explain the experimentally observed instabilities in 3D. While the main ingredients of the instability mechanism are independent of the spatial dimension, there are some important differences between the 2D and 3D systems. In particular, slight buckling of a cylindrical tube does not change its volume. Therefore, at small buckling amplitudes, the wall deformation does not induce any net axial flows – the dominant flows occur in the transverse cross-sections.

THE PROBLEM

We consider the flow through a tube of undeformed radius a whose walls perform small-amplitude, high-frequency oscillations of amplitude $\epsilon^* = \epsilon a$ where $\epsilon \ll 1$. We scale all lengths on the tube radius a and use Lagrangian coordinates (ξ_1, ξ_2) to parametrise the wall shape as

$$\mathbf{R}_w(\xi_1, \xi_2, t) = \mathbf{R}_0(\xi_1, \xi_2) + \epsilon\, \mathbf{v}(\xi_1, \xi_2, t), \tag{1}$$

where the wall displacement $\mathbf{v}(\xi_1, \xi_2, t)$ has a harmonic time-dependence with period T. We assume that a steady pressure gradient drives a steady flow with mean velocity U through the tube. We use a/T to non-dimensionalise the velocities, scale the pressure on the inertial scale and non-dimensionalise time with the period T of the wall oscillation. The fluid flow is then described by the non-dimensional Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{\alpha^2} \nabla^2 \mathbf{u} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\alpha = [\rho a^2/(\mu T)]^{1/2} \gg 1$ is the Womersley number. The no-slip condition requires that

$$\mathbf{u} = \epsilon\, \partial \mathbf{v}/\partial t \quad \text{on the wall.} \tag{3}$$

We decompose all quantities into steady and unsteady contributions and use the mean velocity U to scale the steady velocity; the wall velocity scale ϵ^*/T provides the scale for the unsteady velocity so that

$$\mathbf{u}(x_j, t) = St^{-1} \overline{\mathbf{u}}(x_j) + \epsilon\, \widehat{\mathbf{u}}(x_j, t) \quad \text{and} \quad p(x_j, t) = St^{-2} \overline{p}(x_j) + \epsilon\, \widehat{p}(x_j, t), \tag{4}$$

where $St = a/(UT) \gg 1$ is the Strouhal number. We assume that $\lambda = \epsilon\, St = \mathcal{O}(1)$. In this case the expansion

$$\widehat{\mathbf{u}}(x_j, t) = \widehat{\mathbf{u}}_0(x_j, t) + St^{-1}\, \widehat{\mathbf{u}}_1(x_j, t) + ... \quad \text{and} \quad \widehat{p}(x_j, t) = \widehat{p}_0(x_j, t) + St^{-1}\, \widehat{p}_1(x_j, t) + ... \tag{5}$$

shows that, for large St, the leading-order oscillatory flow $(\widehat{\mathbf{u}}_0, \widehat{p}_0)$ is governed by

$$\frac{\partial \widehat{\mathbf{u}}_0}{\partial t} = -\nabla \widehat{p}_0 + \frac{1}{\alpha^2}\nabla^2 \widehat{\mathbf{u}}_0 \quad \text{and} \quad \nabla \cdot \widehat{\mathbf{u}}_0 = 0, \quad \text{subject to} \quad \widehat{\mathbf{u}}_0 = \partial \mathbf{v}/\partial t \quad \text{on the wall.} \tag{6}$$

Hence, for $\alpha \gg 1$, the flow consists of an inviscid core flow, governed by $\partial \widehat{\mathbf{u}}_0/\partial t = -\nabla \widehat{p}_0$, with Stokes (boundary) layers of thickness $\delta \propto \alpha^{-1}$ on the tube walls. If the wall performs small-amplitude oscillations in one of its fundamental modes, the tube's cross sectional area remains constant and the in-plane divergence of the transverse velocities has a zero cross-sectional average.

In this paper, we make the plausible assumption that the axial component of the leading order velocity perturbation $\widehat{\mathbf{u}}_0$ is identically equal to zero. Since, at large St, the leading-order unsteady perturbation is independent of the steady through flow, we consider the two-dimensional model problem, sketched in Fig. 1, with a combination of numerical and asymptotic methods. A fully-adaptive finite element code is used to determine the dependence of the flowfield on the parameters. The numerical simulations are complemented by analytical studies which are based on the perturbation expansion (5) and provide explicit predictions for the viscous dissipation at large α.

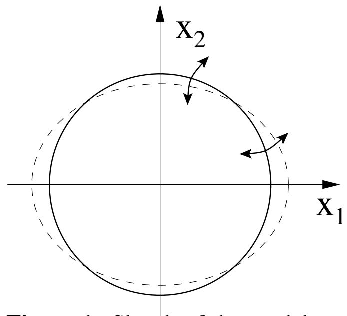

Figure 1. Sketch of the model problem: The vibrating tube wall induces a transverse oscillatory flow.

RESULTS

In many previous studies (e.g. [3]), it had been assumed that the wall oscillation deforms the tube into an elliptical shape. In this case the flow in the core is an unsteady stagnation point flow, $\mathbf{u}_{stag}$, which happens to satisfy the no-slip condition (3). Therefore, $\widehat{\mathbf{u}}_0 \equiv \mathbf{u}_{stag}$ in the entire domain and no boundary layers develop on the wall. The viscous dissipation is spatially uniform and independent of α, and it remains relatively small throughout the oscillation; see the solid line in Fig. 2.

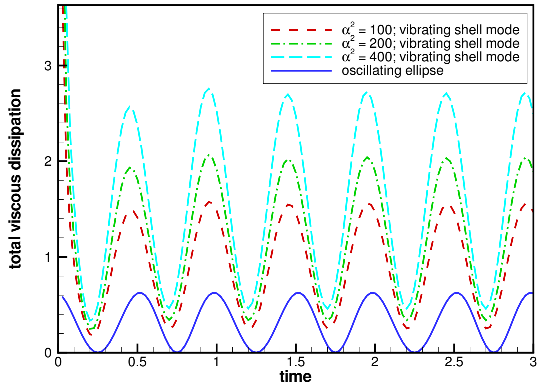

Figure 2. Total viscous dissipation for ring oscillations in different modes and at different Womersley numbers. $\epsilon = 0.1$.

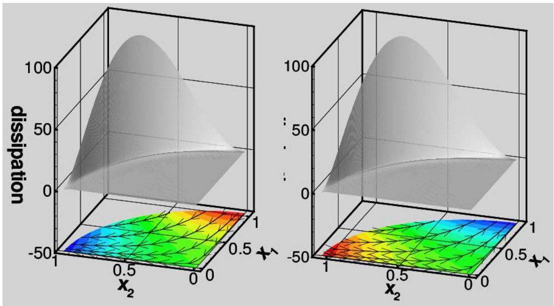

Figure 3. Streamlines and colour pressure contours (plotted in the x_1-x_2-plane) and the local dissipation during two instants of the oscillation. $\alpha^2 = 400$, $\epsilon = 0.1$. Only one quarter of the domain is shown.

If the wall displacement field represents the eigenmodes of a vibrating elastic shell, the core flow is still given by $\mathbf{u}_{stag}$, but in this case $\mathbf{u}_{stag}$ does not satisfy no-slip condition (3). This causes the formation of boundary layers which are illustrated in Fig. 3. The viscous dissipation in the boundary layers is very large and dominates the total viscous dissipation in the domain as indicated by the broken lines in Fig. 2.

CONCLUSIONS AND FURTHER WORK

The velocity perturbations induced by small-amplitude, high-frequency vibrations of a fluid-conveying cylindrical tube are dominated by their transverse components. The total viscous dissipation, whose magnitude is likely to be one of the factors that control the onset of self-excited oscillations in 3D, depends sensitively on the mode shapes of the wall oscillation. In the final part of the talk we will present preliminary results from an extended analysis which includes fluid-structure interaction and therefore allows the prediction of the Womersley number of the oscillation as a function of the remaining system parameters.

References

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[2] Jensen, O.E. & Heil, M. (2003) High-frequency self-excited oscillations in a collapsible-channel flow. J. Fluid Mech. **481**, 235-268.

[3] Padmanabhan, N. & Pedley, T.J. (1987) Three-dimensional steady streaming in a uniform tube with an oscillating elliptical cross-section. J. Fluid Mech. **178**, 325-343.