Annex 4
Linear temporal stability analysis of a liquid filament in a laminar pipe flow of another immiscible liquid

by

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1. Introduction

In emulsification processes, various techniques are in use for dispersing one liquid in another, immiscible liquid. Besides techniques relying on stirring, the action of ultrasound, and the effects of a turbulent flow of the two-phase mixture through a narrow gap, an important technique relies on the break-up of a thread of one liquid in a confined coaxial flow with the other liquid as the surrounding carrier fluid. Usually, an arrangement of many liquid jets in the carrier flow is produced by an array of nozzle holes. The geometry with a single jet is arrangement of a liquid cylinder in a confined coaxial pipe flow, so that the ambient flow does not extend to infinity. The jet breaks down due to its capillary instability and forms the droplets to be emulsified. The present paper undertakes a linear temporal stability analysis of this flow and derives its dispersion relation. Predictions about the most probable drop size derived from the dispersion relation at the maximum growth rate of axisymmetric disturbances are compared with experimental data from the literature.

2. Description of the undisturbed coaxial flow

The coaxial flow situation is described by the sketch in Fig. 1, which defines the properties of the inner and outer flows of the two liquids. The outer flow is the carrier liquid, and the inner one the liquid to be turned into droplets in order to be emulsified. The whole process takes place in a confined flow situation with the inner radius R of the tube, inside which the carrier liquid and a jet of the liquid to be emulsified move. As an alternative, one could certainly consider a case where the jet of liquid to be emulsified is ejected into an infinite bath of the carrier liquid. This is the case considered in [1], but using the method of viscous potential flow.

In the figure, the liquids flow from left to right due to an applied pressure difference per unit length down the tube. The spaces they occupy are the cylinder of radius a for the inner, and the annular space $a \le r \le R$ for the outer liquid. Both liquids are treated as incompressible and Newtonian. The effects of body forces are neglected. The flow is considered as steady, hydraulically developed, and axisymmetric. The pressure drop is the same for both the inner and the outer liquid, since without body forces the pressure in hydraulically developed flow can only be the same for both the inner and the outer flows – except a constant pedestal which is due to the curved surface of the jet of inner liquid and does not change the pressure gradient in the axial direction.

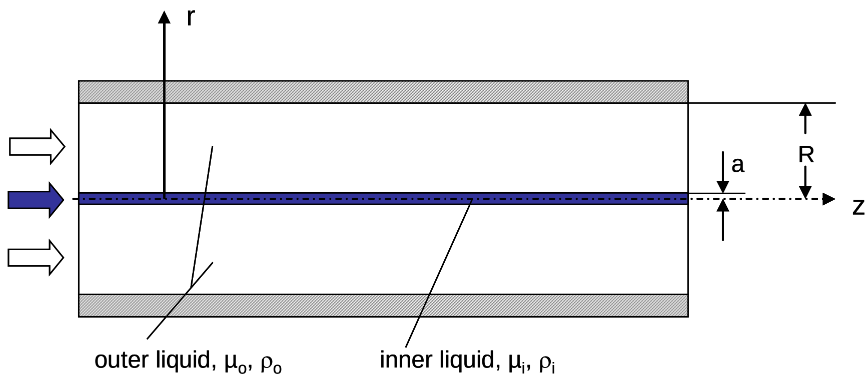

Fig. 1: Coaxial two-component liquid flow in a confined geometry.

The flow is described by the equations of motion in cylindrical coordinates for incompressible Newtonian fluids, which are not detailed here. For the hydraulically developed, steady and axisymmetric flow, the only velocity component that does not vanish identically is the axial component U_z (in the z direction). The differential equation which governs the velocity profiles $U_z(r)$ for both the inner and the outer flows reads

$$0 = -\frac{dp}{dz} + \mu \frac{1}{r}\frac{d}{dr}\left(r \frac{dU_z(r)}{dr} \right) \quad . \tag{1}$$

The general solution is

$$U_z(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 + C \ln r + D \quad . \tag{2}$$

where C and D are constants which are to be determined by the boundary conditions of the flow. For distinguishing between the two flow regimes, we denote the inner flow with subscript i, and the outer flow with subscript o. The total number of unknown constants in the two velocity profiles $U_{z,i}(r)$ and $U_{z,o}(r)$ for the inner and outer flows, respectively, is five – four constants C and D, and the radius a of the inner jet. The five conditions available for determining the five values read

1. $\left. \frac{dU_{z,i}}{dr} \right|_{r=0} = 0$ regularity of the solution on the axis; (3a)
2. $\mu_i \left. \frac{dU_{z,i}}{dr} \right|_{r=a} = \mu_o \left. \frac{dU_{z,o}}{dr} \right|_{r=a}$ shear stress continuous at the interface; (3b)
3. $\left. U_{z,o} \right|_{r=R} = 0$ no-slip condition at the inner wall of the tube; (3c)
4. $\left. U_{z,i} \right|_{r=a} = \left. U_{z,o} \right|_{r=a}$ no-slip condition at the interface; (3d)
5. $\int_{r=0}^{a} U_{z,i} 2\pi r dr = \dot{V}_i$ inner volume flow rate for determining a. (3e)

Application of the first four conditions to the general solution (2) of the equation of motion (1) yields the constants $C_o = C_i = 0$,

$$D_o = -\frac{1}{4\mu_o}\frac{dp}{dz} R^2 \quad , \tag{4a}$$

and

$$D_i = -\frac{1}{4\mu_i}\frac{dp}{dz}a^2\left[1+\frac{\mu_i}{\mu_o}\left(\frac{R^2}{a^2}-1\right)\right] \quad . \tag{4b}$$

Denoting the viscosity ratio $\mu_i/\mu_o{:=}\varepsilon$, the two velocity profiles read

$$U_{z,i}(r) = -\frac{1}{4\mu_i}\frac{dp}{dz}R^2\left[\frac{a^2}{R^2}+\varepsilon\left(1-\frac{a^2}{R^2}\right)-\frac{r^2}{R^2}\right] \tag{5a}$$

and

$$U_{z,o}(r) = -\frac{1}{4\mu_o}\frac{dp}{dz}R^2\left(1-\frac{r^2}{R^2}\right) \tag{5b}$$

for the inner and outer flows, respectively, which we will represent by the form $U_z(r)=A+Br^2$ later. For determining the radius a of the inner liquid flow, the volume flow rates of the outer and the inner flows can be determined by integration of the velocity profiles. The result for the inner flow is

$$\dot{V}_i = -\frac{\pi}{2\mu_i}\frac{dp}{dz}a^2\int_{r=0}^{a}\left[1+\frac{\mu_i}{\mu_o}\left(\frac{R^2}{a^2}-1\right)-\frac{r^2}{a^2}\right]r\,dr = -\frac{\pi}{8\mu_i}\frac{dp}{dz}R^4\left[\frac{a^4}{R^4}+2\frac{\mu_i}{\mu_o}\left(1-\frac{a^2}{R^2}\right)\frac{a^2}{R^2}\right] \quad . \tag{6}$$

The same calculation can also be carried out for the outer fluid, where the integration of the velocity profile over the domain $a \le r \le R$ must yield the volume flow rate of the outer liquid as

$$\dot{V}_o = -\frac{\pi}{2\mu_o}\frac{dp}{dz}R^2\int_{r=a}^{R}\left[1-\frac{r^2}{R^2}\right]r\,dr = -\frac{\pi}{8\mu_o}\frac{dp}{dz}R^4\left[1-2\frac{a^2}{R^2}\left(1-\frac{a^2}{2R^2}\right)\right] \quad . \tag{7}$$

Calculating the ratio of these two equations, and denoting the ratio of the volume flow rates $\dot{V}_i/\dot{V}_o := \varphi$, we obtain for the ratio of the radii of the filament and of the tube

$$\frac{a}{R} = \left[\frac{\sqrt{1+\varphi/\varepsilon}-(1+\varphi)}{1/\varepsilon-(2+\varphi)}\right]^{1/2} \tag{8}$$

This equation yields the correct asymptotic behaviour, i.e. $a/R\to 1$ for $\varphi\to\infty$, and $a/R\to 0$ for $\varphi\to 0$. Also, for $\varepsilon=1$ the sum of the two flow rates yields the flow rate of the Hagen-Poiseuille flow in the tube with radius R. For the later calculation, the ratio of the volume flow rates of liquid $\dot{V}_i$ of the inner flow and $\dot{V}_o$ of the outer flow and the viscosity ratio of the two fluids will be taken as known and the jet radius a determined using (8). Experimental values for realistic situations may, e.g., be taken from [2].

3. Linear stability analysis

Our analysis considers the linear temporal instability of the coaxial flow against axisymmetric disturbances. A similar analysis, but for different flow geometry, was carried out by the authors of [1], who derived a dispersion relation for the flow. Also, they did not derive information about a drop size from this kind of an analysis. Also, they used the concept of viscous potential flow with viscous pressure correction, according to [3].

For analysing the behaviour of a disturbance of the base flow we linearise the momentum equations. Using the decomposition of the radial and axial velocities into a base flow part (upper-case U) and a disturbance part (lower-case u), the linearised momentum equations for an incompressible Newtonian fluid in cylindrical coordinates are (subscript r – radial, subscript z – axial velocity component) read

r component
$$\frac{\partial u_r}{\partial t} + U_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\partial^2 u_r}{\partial z^2}\right] \tag{9a}$$

and

z component
$$\frac{\partial u_z}{\partial t} + u_r \frac{dU_z}{dr} + U_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right] \quad . \tag{9b}$$

In these equations, the pressure p and the lower-case velocities are due to the disturbance. Equations (9a) and (9b) must later be formulated for the inner and outer flows separately for analysing the disturbances. Accordingly, for the two different flow regions, the related undisturbed base-flow velocities U_z must be taken. The latter are given by Eqs. (5a) and (5b) above for the inner and outer flows, respectively.

The continuity equation for the axisymmetric disturbance in cylindrical coordinates reads

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0 \quad . \tag{10}$$

The approach to a solution of the set of equations (9)–(10) is to first formulate the disturbance velocities in the radial and the axial directions using the Stokes stream function ψ as per

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \qquad \text{and} \qquad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} \quad , \tag{11a,b}$$

which ensures that the continuity equation (10) is satisfied automatically. For the stream function we get from the momentum equations that it must be of the form

$$\psi = \Psi(r)\cdot \exp(ikz + \omega t) \quad . \tag{12}$$

In this equation, k is the (real) dimensional wave number, and $\omega = \omega_r + i\omega_i$ is a complex frequency, where ω_r may be interpreted as a growth rate of the disturbances, and ω_i as the oscillation frequency. Eliminating the pressure from the momentum equations (9a) and (9b) and substituting the definitions (11a,b) into the resultant equation, we obtain the relation

$$\left(\frac{\partial}{\partial t} + U_z\frac{\partial}{\partial z} - \nu D\right)D\psi = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{dU_z}{dr}\right)\frac{\partial \psi}{\partial z} \quad , \tag{13}$$

where the operator D is

$$D = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad . \tag{14}$$

The differential equation (13) for the stream function differs from the corresponding one obtained by Tomotika [4] in that, on the left-hand side, a substantial derivative of $D\psi$ occurs rather than a derivative w.r.t. time alone, and that the right-hand side is not zero at the first glance. When examining the term on the right in (13), however, we see that, for a parabolic velocity profile of the form $U_z(r)=A+Br^2$, as found in the present investigations, the term vanishes, since the argument of the partial derivative w.r.t. r is a constant. So the differential equation (13) is in fact

$$\left(\frac{\partial}{\partial t}+U_z\frac{\partial}{\partial z}-\nu D\right)D\psi=0 \quad . \tag{15}$$

For this equation, the argumentation of Tomotika, saying that the operator in round brackets and D are commutative, still holds. We can therefore compose the stream function of two components, $\psi = \psi_1 + \psi_2 = (\Psi_1+\Psi_2)\cdot\exp(ikz+\omega t)$, which are given by the two differential equations

$$\begin{gathered} D\psi_1=0 \\ D\psi_2-\frac{1}{\nu}\left(\frac{\partial\psi_2}{\partial t}+U_z\frac{\partial\psi_2}{\partial z}\right)=0 \end{gathered} \tag{16}$$

Following the structure (12) of the stream functions, which holds for ψ_1 and ψ_2 also, we obtain for the amplitude functions Ψ_1 and Ψ_2 the two ordinary differential equations

$$r^2\Psi_1''-r\Psi_1'-k^2r^2\Psi_1=0 \tag{17a}$$

and

$$r^2\Psi_2''-r\Psi_2'-r^2\left(k^2+\frac{\omega+ikU_z(r)}{\nu}\right)\Psi_2=0 \quad , \tag{17b}$$

respectively, where the primes indicate derivatives w.r.t. the radial coordinate r. Equation (17a) is a Bessel differential equation, for which we obtain, in the same manner as in [4], the solution

$$\psi_1(r,z,t)=\left[A_1rI_1(kr)+A_2rK_1(kr)\right]\cdot\exp(ikz+\omega t) \quad . \tag{18}$$

The solution of the second ODE (17b) requires some more care. The equation corresponds to the form

$$x^2y''+\left(a\,x^n+b\right)xy'+\left(\alpha\,x^{2n}+\beta\,x^n+\gamma\right)y=0 \tag{19}$$

given as type 2.215 in [5], with a= 0, b= –1, n= 2, $\alpha= -ikB/\nu$, $\beta= -k^2- (\omega+ikA)/\nu$, and $\gamma= 0$, for the velocity profile $U_z(r)$ given as $U_z(r)= A+Br^2$ for our present parabolic base flow velocity distributions. In (19), x corresponds to r, and y to Ψ_2. First a transformation to a simpler form of the ODE using

$$\Psi_2(r)=\xi^j\cdot\eta(\xi) \quad \text{with} \quad \xi= r^n= r^2\text{, where j is a solution of } 4j^2 - 4j= 0 \tag{20}$$

leads us to two differential equations, one for each value j= 0 and j= 1 obtained from (20). The transformed differential equations read

$$(j=0) \qquad \xi\eta''+\left(-\frac{ikB}{4\nu}\xi-\frac{k^2}{4}-\frac{\omega+ikA}{4\nu}\right)\eta=0 \tag{21a}$$

$$(j=1)\qquad \xi\eta''+2\eta'+\left(-\frac{ikB}{4\nu}\xi-\frac{k^2}{4}-\frac{\omega+ikA}{4\nu}\right)\eta=0\ . \tag{21b}$$

We first look at the first equation (21a) (for j= 0), which may be transformed by $x_1=\sqrt{-ikB/4\nu}\,\xi,\ y_1(x_1)=\eta(\xi)$ into the form

$$x_1 y_1''+\left(x_1+\frac{\left(-\frac{k^2}{4}-\frac{\omega+ikA}{4\nu}\right)}{\sqrt{-\frac{ikB}{4\nu}}}\right)y_1=0\ , \tag{22}$$

which is of type 2.92 in [5]. From the information about ODE type 2.134 in [5] we find that (22) may be transformed by $x_2=2ix_1$, $y_2(x_2)=y_1(x_1)$ into the form

$$4x_2 y_2''-\left(x_2-i\frac{\left(k^2+(\omega+ikA)/\nu\right)}{\sqrt{-ikB/\nu}}\right)y_2=0\ , \tag{23}$$

from where the transformation $y_2(x_2)=x_2\exp(-x_2/2)\,u(x_2)$ leads us to the differential equation

$$x_2u''+(2-x_2)u'-\left(1-\frac{i}{4}\frac{k^2+(\omega+ikA)/\nu}{\sqrt{-ikB/\nu}}\right)u=0\ . \tag{24}$$

This equation is easily recognized as the confluent hypergeometric differential equation in the function $u(x_2)$. The general solution of this equation is

$$u(x_2)=C_1M\left(1-\frac{i}{4}\frac{k^2+(\omega+ikA)/\nu}{\sqrt{-ikB/\nu}},2,x_2\right)+C_2U\left(1-\frac{i}{4}\frac{k^2+(\omega+ikA)/\nu}{\sqrt{-ikB/\nu}},2,x_2\right), \tag{25}$$

where the functions M and U are confluent hypergeometric functions of the first and second kinds, respectively [6]. The former is also called Kummer's function. In this solution we must discard the function U, which is undefined for the value of 2 of the second argument, by setting $C_2=0$. The various transformations back to the original equation (17b) lead to the final formulation of its general solution, which finally reads

$$\Psi_{2,0}(r)=C_0\sqrt{\frac{ikB}{\nu}}r^2\exp\left(r^2\sqrt{ikB/(4\nu)}\right)\cdot M\left(1+\frac{1}{4}\sqrt{i}\frac{k^2+(\omega+ikA)/\nu}{\sqrt{kB/\nu}},2,-\sqrt{\frac{ikB}{\nu}}r^2\right)\ , \tag{26}$$

where C_0 is an arbitrary constant. When solving the ODE (21b) for j= 1 above in the same way, we see that, after discarding undefined parts of the solution, we obtain the result

$$\Psi_{2,1}(r)=C_1\sqrt{\frac{ikB}{\nu}}r^2\exp\left(-r^2\sqrt{ikB/(4\nu)}\right)\cdot M\left(1-\frac{1}{4}\sqrt{-i}\frac{k^2+(\omega+ikA)/\nu}{\sqrt{kB/\nu}},2,\sqrt{\frac{ikB}{\nu}}r^2\right)\ , \tag{27}$$

with an arbitrary constant C_1. This finalises the solution of the differential equation (15) for the stream function. The general solution is obtained from (18), (26), and (27) as

$$\psi(r,z,t)=\Big[C_1 r I_1(kr)+C_2 r K_1(kr)+$$
$$+C_3\sqrt{\frac{ikB}{\nu}}r^2\exp\left(r^2\sqrt{ikB/(4\nu)}\right)\cdot M\left(1+\frac{1}{4}\sqrt{i}\,\frac{k^2+(\omega+ikA)/\nu}{\sqrt{kB/\nu}},2,-\sqrt{\frac{ikB}{\nu}}r^2\right)+$$
$$+C_4\sqrt{\frac{ikB}{\nu}}r^2\exp\left(-r^2\sqrt{ikB/(4\nu)}\right)\cdot M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA)/\nu}{\sqrt{kB/\nu}},2,\sqrt{\frac{ikB}{\nu}}r^2\right)\Bigg]\cdot\exp(ikz+\omega t) \quad (28)$$

This equation corresponds to Tomotika's solution (15) in his paper [4]. The stream function (28) must be formulated for the inner and outer cylindrical flows. The functions read

$$\psi_i(r,z,t)=\Big[C_{1,i} r I_1(kr)+$$
$$+C_{3,i}\sqrt{\frac{ikB_i}{\nu_i}}r^2\exp\left(r^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot M\left(1+\frac{1}{4}\sqrt{i}\,\frac{k^2+(\omega+ikA_i)/\nu_i}{\sqrt{kB_i/\nu_i}},2,-\sqrt{\frac{ikB_i}{\nu_i}}r^2\right)+$$
$$+C_{4,i}\sqrt{\frac{ikB_i}{\nu_i}}r^2\exp\left(-r^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA_i)/\nu_i}{\sqrt{kB_i/\nu_i}},2,\sqrt{\frac{ikB_i}{\nu_i}}r^2\right)\Bigg]\cdot\exp(ikz+\omega t) \quad (29a)$$

and

$$\psi_o(r,z,t)=\Big[C_{1,o} r I_1(kr)+C_{2,o} r K_1(kr)+$$
$$+C_{3,o}\sqrt{\frac{ikB_o}{\nu_o}}r^2\exp\left(r^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M\left(1+\frac{1}{4}\sqrt{i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,-\sqrt{\frac{ikB_o}{\nu_o}}r^2\right)+$$
$$+C_{4,o}\sqrt{\frac{ikB_o}{\nu_o}}r^2\exp\left(-r^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,\sqrt{\frac{ikB_o}{\nu_o}}r^2\right)\Bigg]\cdot\exp(ikz+\omega t)\;, \quad (29b)$$

respectively, where the function K_1, which diverges for $r\to 0$, was discarded from the inner function ψ_i by setting $C_{2,i} = 0$. The constants A and B for the inner and outer flows are

$$A_i=\frac{2}{\pi}\frac{\dot{V}_o}{R^2}\frac{\frac{a^2}{R^2}+\varepsilon\left(1-\frac{a^2}{R^2}\right)}{\varepsilon\left[1-2\frac{a^2}{R^2}\left(1-\frac{a^2}{2R^2}\right)\right]}\,,\qquad B_i=-\frac{2}{\pi}\frac{\dot{V}_o}{R^4}\frac{1}{\varepsilon\left[1-2\frac{a^2}{R^2}\left(1-\frac{a^2}{2R^2}\right)\right]} \quad (30a)$$

$$A_o=\frac{2}{\pi}\frac{\dot{V}_o}{R^2}\frac{1}{1-2\frac{a^2}{R^2}\left(1-\frac{a^2}{2R^2}\right)},\qquad B_o=-\frac{2}{\pi}\frac{\dot{V}_o}{R^4}\frac{1}{1-2\frac{a^2}{R^2}\left(1-\frac{a^2}{2R^2}\right)}\;. \quad (30b)$$

3.1 Boundary conditions

We now use the stream functions (30a) and (30b) for formulating the boundary conditions available for calculating the seven integration constants $C_{1,i}$ – $C_{4,o}$. The seven conditions are

1. no slip at the tube wall r=R ($u_{r,o}(r{=}R)= 0$, $u_{z,o}(r{=}R)= 0$ – two conditions); (31a)
2. no slip at the interface of the two flows ($u_{r,i}(r{=}a)= u_{r,o}(r{=}a)$, $u_{z,i}(r{=}a)= u_{z,o}(r{=}a)$ – two conditions); (31b)

3. tangential stress on the interface is continuous at the interface r=a; (31c)
4. the difference in the normal stress between the inner and outer flows at the interface r=a is due to the interfacial tension σ; (31d)
5. regularity of the right-hand sides of the momentum equations (10a) and (10b) on the tube axis r=0. (31e)

We first look at the *first boundary condition* (31a). It requires that the disturbance velocities, in the same manner as the velocities of the undisturbed base flow, vanish at the tube wall. The related equations read

$$u_{r,o}\Big|_{r=R} = \frac{1}{r}\frac{\partial \psi_o}{\partial z}\Big|_{r=R} = 0 \text{ and } u_{z,o}\Big|_{r=R} = -\frac{1}{r}\frac{\partial \psi_o}{\partial r}\Big|_{r=R} = 0 \quad , \tag{32a,b}$$

and we obtain from them the requirements that

$$\begin{aligned} & C_{1,o} I_1(kR) + C_{2,o} K_1(kR) + \\ & + C_{3,o}\sqrt{\frac{ikB_o}{\nu_o}} R \exp\left(R^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M_{1oR} + C_{4,o}\sqrt{\frac{ikB_o}{\nu_o}} R \exp\left(-R^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M_{2oR} = 0 \end{aligned} \tag{33a}$$

and

$$\begin{aligned} & C_{1,o} k I_0(kR) - C_{2,o} k K_0(kR) + \\ & + 2C_{3,o}\sqrt{\frac{ikB_o}{\nu_o}} \exp\left(R^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot\left[M_{1oR} + \sqrt{\frac{ikB_o}{\nu_o}} R^2\left(\frac{1}{2}M_{1oR} - M'_{1oR}\right)\right] + \\ & + 2C_{4,o}\sqrt{\frac{ikB_o}{\nu_o}} \exp\left(-R^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot\left[M_{2oR} - \sqrt{\frac{ikB_o}{\nu_o}} R^2\left(\frac{1}{2}M_{2oR} - M'_{2oR}\right)\right] = 0 \end{aligned} \tag{33b}$$

In these equations we have set

$$M_{1oR} = M\left(1 + \frac{1}{4}\sqrt{i}\,\frac{k^2 + (\omega + ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}}, 2, -\sqrt{\frac{ikB_o}{\nu_o}} R^2\right) \quad \text{and} \tag{34a}$$

$$M_{2oR} = M\left(1 - \frac{1}{4}\sqrt{-i}\,\frac{k^2 + (\omega + ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}}, 2, \sqrt{\frac{ikB_o}{\nu_o}} R^2\right) \quad . \tag{34b}$$

The prime at the M indicates the first derivative of Kummer's function w.r.t. the third argument.

The *second boundary condition* (31b) states that the velocities of the inner and outer flows are equal at the interface r=a, i.e., that

$$u_{r,i}\Big|_{r=a} = u_{r,o}\Big|_{r=a} \quad \text{and} \quad u_{z,i}\Big|_{r=a} = u_{z,o}\Big|_{r=a} \quad . \tag{35a,b}$$

This is equivalent to the requirements that

$$\left.\frac{\partial\psi_i}{\partial z}\right|_{r=a} = \left.\frac{\partial\psi_o}{\partial z}\right|_{r=a} \quad \text{and} \quad \left.\frac{\partial\psi_i}{\partial r}\right|_{r=a} = \left.\frac{\partial\psi_o}{\partial r}\right|_{r=a} \quad . \tag{36a,b}$$

This requirement leads to the two equations

$$\begin{aligned}
&C_{1,i} I_1(ka) + C_{3,i}\sqrt{\frac{ikB_i}{\nu_i}}\, a \exp\left(a^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot M_{1ia} + \\
&+ C_{4,i}\sqrt{\frac{ikB_i}{\nu_i}}\, a \exp\left(-a^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot M_{2ia} = \\
&C_{1,o} I_1(ka) + C_{2,o} K_1(ka) + \\
&+ C_{3,o}\sqrt{\frac{ikB_o}{\nu_o}}\, a \exp\left(a^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M_{1oa} + \\
&+ C_{4,o}\sqrt{\frac{ikB_o}{\nu_o}}\, a \exp\left(-a^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M_{2oa}
\end{aligned} \tag{37a}$$

and

$$\begin{aligned}
&C_{1,i} k I_0(ka) + 2C_{3,i}\sqrt{\frac{ikB_i}{\nu_i}} \exp\left(a^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot\left[M_{1ia} + \sqrt{\frac{ikB_i}{\nu_i}}\, a^2\left(\frac{1}{2}M_{1ia} - M'_{1ia}\right)\right] + \\
&+ 2C_{4,i}\sqrt{\frac{ikB_i}{\nu_i}} \exp\left(-a^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot\left[M_{2ia} - \sqrt{\frac{ikB_i}{\nu_i}}\, a^2\left(\frac{1}{2}M_{2ia} - M'_{2ia}\right)\right] = \\
&C_{1,o} k I_0(ka) - C_{2,o} k K_0(ka) + \\
&+ 2C_{3,o}\sqrt{\frac{ikB_o}{\nu_o}} \exp\left(a^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot\left[M_{1oa} + \sqrt{\frac{ikB_o}{\nu_o}}\, a^2\left(\frac{1}{2}M_{1oa} - M'_{1oa}\right)\right] + \\
&+ 2C_{4,o}\sqrt{\frac{ikB_o}{\nu_o}} \exp\left(-a^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot\left[M_{2oa} - \sqrt{\frac{ikB_o}{\nu_o}}\, a^2\left(\frac{1}{2}M_{2oa} - M'_{2oa}\right)\right]
\end{aligned} \tag{37b}$$

In these equations we have used the abbreviations

$$M_{1ia} = M\left(1 + \frac{1}{4}\sqrt{i}\,\frac{k^2 + (\omega + ikA_i)/\nu_i}{\sqrt{kB_i/\nu_i}}, 2, -\sqrt{\frac{ikB_i}{\nu_i}}\, a^2\right) \tag{38a}$$

$$M_{2ia} = M\left(1 - \frac{1}{4}\sqrt{-i}\,\frac{k^2 + (\omega + ikA_i)/\nu_i}{\sqrt{kB_i/\nu_i}}, 2, \sqrt{\frac{ikB_i}{\nu_i}}\, a^2\right) \quad , \tag{38b}$$

$$M_{1oa} = M\left(1 + \frac{1}{4}\sqrt{i}\,\frac{k^2 + (\omega + ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}}, 2, -\sqrt{\frac{ikB_o}{\nu_o}}\, a^2\right) \quad , \tag{38c}$$

$$M_{2oa} = M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,\sqrt{\frac{ikB_o}{\nu_o}}a^2\right) \quad , \tag{38d}$$

and the prime at the symbol M again indicates the first derivative of Kummer's function w.r.t. the third argument.

The *third boundary condition* (31c) states that the tangential stress be continuous across the interface r=a, i.e. that

$$\mu_i\left(\frac{\partial^2\psi_i}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_i}{\partial r}-\frac{\partial^2\psi_i}{\partial z^2}\right)\Bigg|_{r=a} = \mu_o\left(\frac{\partial^2\psi_o}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_o}{\partial r}-\frac{\partial^2\psi_o}{\partial z^2}\right)\Bigg|_{r=a} \quad . \tag{39}$$

The expressions in the brackets may be rewritten and expanded to yield

$$\begin{aligned}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial z^2}\right) = \Bigg(&\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-k^2\psi_1+2k^2\psi_1+\\ &+\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}-k^2\psi_2-\frac{\omega+ikU_z(r)}{\nu}\psi_2+2k^2\psi_2+\frac{\omega+ikU_z(r)}{\nu}\psi_2\Bigg)\end{aligned} , \tag{40}$$

which enables us to make use of the differential equations (17a) and (17b) for simplifying them. We obtain

$$\mu_i\left(2k^2\Psi_i+\frac{\omega+ikU_{z,i}(r)}{\nu_i}\Psi_{2,i}\right)\Bigg|_{r=a} = \mu_o\left(2k^2\Psi_o+\frac{\omega+ikU_{z,o}(r)}{\nu_o}\Psi_{2,o}\right)\Bigg|_{r=a} \quad . \tag{41}$$

Substituting the results for the amplitude functions Ψ in (29a) and (29b) into (41), we obtain

$$\begin{aligned}&\mu_i\Bigg[2k^2C_{1,i}I_1(ka)+C_{3,i}\sqrt{\frac{ikB_i}{\nu_i}}a\exp\left(a^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot M_{1ia}\cdot\left(2k^2+\frac{\omega+ikU_{z,i}(a)}{\nu_i}\right)+\\ &+C_{4,i}\sqrt{\frac{ikB_i}{\nu_i}}a\exp\left(-a^2\sqrt{ikB_i/(4\nu_i)}\right)\cdot M_{2ia}\cdot\left(2k^2+\frac{\omega+ikU_{z,i}(a)}{\nu_i}\right)\Bigg]=\\ &\mu_o\Bigg[2k^2C_{1,o}I_1(ka)+2k^2C_{2,o}K_1(ka)+C_{3,o}\sqrt{\frac{ikB_o}{\nu_o}}a\exp\left(a^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M_{1oa}\cdot\left(2k^2+\frac{\omega+ikU_{z,o}(a)}{\nu_o}\right)+\\ &+C_{4,o}\sqrt{\frac{ikB_o}{\nu_o}}a\exp\left(-a^2\sqrt{ikB_o/(4\nu_o)}\right)\cdot M_{2oa}\cdot\left(2k^2+\frac{\omega+ikU_{z,o}(a)}{\nu_o}\right)\Bigg]\end{aligned} \tag{42}$$

The *fourth boundary condition* (31d) quantifies the difference of the normal stresses at the inner and outer sides of the interface r=a, which is due to the action of the interfacial tension between the two fluids. The condition says that the radial stresses Σ_{rr} at the two sides of the cylinder r=a are related as

$$\Sigma_{rr,i}\big|_{r=a} = \Sigma_{rr,o}\big|_{r=a}+p_\sigma \quad \text{with the capillary pressure} \quad p_\sigma = \frac{\sigma}{a^2}\left(1-k^2a^2\right)\cdot\xi \quad , \tag{43}$$

where ξ is the radial displacement of the interface between the inner and outer flows, which is given as

$$\xi = \int u_{r,i}\big|_{r=a} dt = \int \frac{1}{a}\frac{\partial \psi}{\partial z}\bigg|_{r=a} dt = \frac{ik}{\omega a}\psi_i\big|_{r=a} \quad . \tag{44}$$

The capillary pressure may therefore be written as

$$p_\sigma = \frac{ik\sigma}{\omega a^3}\left(1-k^2a^2\right)\cdot \psi_i\big|_{r=a} \quad . \tag{45}$$

The radial stress is given as $\Sigma_{rr}= -p+2\mu\partial u_r/\partial r$, where the pressure p is obtained by integration of the momentum equation (9b) and reads

$$p = \frac{\rho}{ikr}\frac{\partial}{\partial r}\left[\left(\omega + ikU_z(r)\right)\psi_1\right] - 4B\rho\psi \quad . \tag{46}$$

The equation resulting from the boundary condition (31d) reads

$$\begin{aligned} &\mu_i\left[2ik\frac{\partial}{\partial r}\left(\frac{\psi_i}{r}\right) + \frac{4B_i}{\nu_i}\psi_i - \frac{1}{ikr}\frac{\partial}{\partial r}\left(\frac{\omega+ikU_{z,i}(r)}{\nu_i}\psi_{1,i}\right)\right]_{r=a} = \\ &\mu_o\left[2ik\frac{\partial}{\partial r}\left(\frac{\psi_o}{r}\right) + \frac{4B_o}{\nu_o}\psi_o - \frac{1}{ikr}\frac{\partial}{\partial r}\left(\frac{\omega+ikU_{z,o}(r)}{\nu_o}\psi_{1,o}\right)\right]_{r=a} + \frac{ik\sigma}{\omega a^3}\left(1-k^2a^2\right)\psi_i \end{aligned} \quad , \tag{47}$$

from which form we obtain the sixth condition for the coefficients $C_{1,i} - C_{4,o}$ by substituting the known expressions for the stream functions. The equation finally reads

$$\begin{aligned} &C_{1,i}\varepsilon\left[2ik^2 I_1'(ka) + \frac{B_i a}{\nu_i}\left(2I_1(ka) - ka\,I_0(ka)\right) - \frac{\omega+ikA_i}{i\nu_i}I_0(ka) - \frac{ik\sigma}{\mu_i\omega a^2}\left(1-k^2a^2\right)I_1(ka)\right] + \\ &+C_{3,i}\varepsilon\left[2i\frac{k}{a^2}\left(M_{1ia} + \sqrt{\frac{ikB_i}{\nu_i}}a^2\left(M_{1ia} - 2M'_{1ia}\right)\right) + 4\frac{B_i}{\nu_i}M_{1ia} - \frac{ik\sigma}{\mu_i\omega a^3}\left(1-k^2a^2\right)M_{1ia}\right]\sqrt{\frac{ikB_i}{\nu_i}}a^2 \exp_{+ia} + \\ &+C_{4,i}\varepsilon\left[2i\frac{k}{a^2}\left(M_{2ia} - \sqrt{\frac{ikB_i}{\nu_i}}a^2\left(M_{2ia} - 2M'_{2ia}\right)\right) + 4\frac{B_i}{\nu_i}M_{2ia} - \frac{ik\sigma}{\mu_i\omega a^3}\left(1-k^2a^2\right)M_{2ia}\right]\sqrt{\frac{ikB_i}{\nu_i}}a^2 \exp_{-ia} - \\ &-C_{1,o}\left[2ik^2 I_1'(ka) + \frac{B_o a}{\nu_o}\left(2I_1(ka) - ka\,I_0(ka)\right) - \frac{\omega+ikA_o}{i\nu_o}I_0(ka)\right] - \\ &-C_{2,o}\left[2ik^2 K_1'(ka) + \frac{B_o a}{\nu_o}\left(2K_1(ka) + ka\,K_0(ka)\right) + \frac{\omega+ikA_o}{i\nu_o}K_0(ka)\right] - \\ &-C_{3,o}\left[2i\frac{k}{a^2}\left(M_{1oa} + \sqrt{\frac{ikB_o}{\nu_o}}a^2\left(M_{1oa} - 2M'_{1oa}\right)\right) + 4\frac{B_o}{\nu_o}M_{1oa}\right]\sqrt{\frac{ikB_o}{\nu_o}}a^2 \exp_{+oa} - \\ &-C_{4,o}\left[2i\frac{k}{a^2}\left(M_{2oa} - \sqrt{\frac{ikB_o}{\nu_o}}a^2\left(M_{2oa} - 2M'_{2oa}\right)\right) + 4\frac{B_o}{\nu_o}M_{2oa}\right]\sqrt{\frac{ikB_o}{\nu_o}}a^2 \exp_{-oa} = 0 \end{aligned} \tag{48}$$

Finally remains the *fifth condition* (31e), which is a regularity condition and brings the seventh piece of information for the coefficients C of the stream functions in Equations (29a) and (29b). This condition expresses that the right-hand sides, where derivatives of the disturbance velocities or the velocities themselves are divided by the radial coordinate r, must be regular on the tube axis r=0. This results in the requirements that

$$\lim_{r\to 0}\frac{\partial u_{z,i}}{\partial r}=0 \quad , \quad \lim_{r\to 0}\frac{\partial u_{r,i}}{\partial r}=0 \quad , \quad \lim_{r\to 0}\frac{u_{r,i}}{r^2}=\text{finite} \qquad \text{(49a,b,c)}$$

where $u_{r,i}$ are the radial, and $u_{z,i}$ the axial disturbance velocities of the inner flow. The requirement (49a) for the velocity $u_{z,i}$ is automatically satisfied by the velocity as defined by equation (11b). The two conditions (49b) and (49c) identically lead to the requirement that the derivative of $u_{r,i}$ w.r.t. the radial coordinate be zero at r=0. This requirement is readily expressed by the condition that

$$C_{1,i}\,k\,I_1'(0)+\sqrt{\frac{ikB_i}{\nu_i}}C_{3,i}+\sqrt{\frac{ikB_i}{\nu_i}}C_{4,i}=0 \quad , \qquad (50)$$

which is the seventh and last condition needed for establishing the dispersion relation of the present flow.

3.2 Dispersion relation

The homogeneous system of equations (33a,b), (37a,b), (42), (48), and (50) for the seven integration constants $C_{1,i}$ through $C_{4,o}$, obtained as a result of application of the seven conditions in (30a) through (30e) to the disturbance stream functions (29a) and (29b), exhibits non-trivial solutions if and only if the value of the coefficient determinant vanishes. This condition finally yields the dispersion relation of the system. This condition for the coefficient determinant, after cancellation of some common factors from the second, third, sixth and seventh columns, is given as equation (51) on the following page. In that equation, we have used abbreviations given below the equation in order to keep the handling of the equation feasible.

$$
\begin{vmatrix}
0 & 0 & 0 & I_1(kR) & \dfrac{K_1(kR)}{K_0(ka)} & \dfrac{R}{a}\dfrac{\exp_{+oR}}{\exp_{+oa}}M_{1oR} & \dfrac{R}{a}\dfrac{\exp_{-oR}}{\exp_{-oa}}M_{2oR} \\
0 & 0 & 0 & kaI_0(kR) & -ka\dfrac{K_0(kR)}{K_0(ka)} & 2F_{3,oR}\dfrac{\exp_{+oR}}{\exp_{+oa}} & 2F_{4,oR}\dfrac{\exp_{-oR}}{\exp_{-oa}} \\
I_1(ka) & M_{1ia} & M_{2ia} & -I_1(ka) & -\dfrac{K_1(ka)}{K_0(ka)} & -M_{1oa} & -M_{2oa} \\
kaI_0(ka) & 2F_{3,ia} & 2F_{4,ia} & -kaI_0(ka) & -ka & -2F_{3,oa} & -2F_{4,oa} \\
2k^2a^2\varepsilon I_1(ka) & \varepsilon M_{1ia}k_{1,i} & \varepsilon M_{2ia}k_{1,i} & -2k^2a^2I_1(ka) & -2k^2a^2-\dfrac{K_1(ka)}{K_0(ka)} & -M_{1oa}k_{1,o} & -M_{2oa}k_{1,o} \\
G_{1,i} & G_{3,i} & G_{4,i} & G_{1,o} & G_{2,o} & G_{3,o} & G_{4,o} \\
ka\,I_1'(0) & \exp_{-ia} & \exp_{+ia} & 0 & 0 & 0 & 0
\end{vmatrix} = 0 \qquad (51)
$$

where we have used the following abbreviations:

$$M_{1oR} = M\left(1+\frac{1}{4}\sqrt{i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,-\sqrt{\frac{ikB_o}{\nu_o}}R^2\right) \qquad M_{2oR} = M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,\sqrt{\frac{ikB_o}{\nu_o}}R^2\right)$$

$$M_{1oa} = M\left(1+\frac{1}{4}\sqrt{i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,-\sqrt{\frac{ikB_o}{\nu_o}}a^2\right) \qquad M_{2oa} = M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA_o)/\nu_o}{\sqrt{kB_o/\nu_o}},2,\sqrt{\frac{ikB_o}{\nu_o}}a^2\right)$$

$$M_{1ia} = M\left(1+\frac{1}{4}\sqrt{i}\,\frac{k^2+(\omega+ikA_i)/\nu_i}{\sqrt{kB_i/\nu_i}},2,-\sqrt{\frac{ikB_i}{\nu_i}}a^2\right) \qquad M_{2ia} = M\left(1-\frac{1}{4}\sqrt{-i}\,\frac{k^2+(\omega+ikA_i)/\nu_i}{\sqrt{kB_i/\nu_i}},2,\sqrt{\frac{ikB_i}{\nu_i}}a^2\right)$$

$$\exp_{+oR} = \exp\left(R^2\sqrt{ikB_o/(4\nu_o)}\right), \qquad \exp_{-oR} = \exp\left(-R^2\sqrt{ikB_o/(4\nu_o)}\right),$$
$$\exp_{+oa} = \exp\left(a^2\sqrt{ikB_o/(4\nu_o)}\right), \qquad \exp_{-oa} = \exp\left(-a^2\sqrt{ikB_o/(4\nu_o)}\right).$$
$$\exp_{+ia} = \exp\left(a^2\sqrt{ikB_i/(4\nu_i)}\right), \qquad \exp_{-ia} = \exp\left(-a^2\sqrt{ikB_i/(4\nu_i)}\right),$$

$$F_{3,oR} = M_{1oR} + \sqrt{\frac{ikB_o}{\nu_o}} R^2 \left(\frac{1}{2} M_{1oR} - M'_{1oR} \right), \quad F_{4,oR} = M_{2oR} - \sqrt{\frac{ikB_o}{\nu_o}} R^2 \left(\frac{1}{2} M_{2oR} - M'_{2oR} \right),$$

$$F_{3,ia} = M_{1ia} + \sqrt{\frac{ikB_i}{\nu_i}} a^2 \left(\frac{1}{2} M_{1ia} - M'_{1ia} \right), \qquad F_{4,ia} = M_{2ia} - \sqrt{\frac{ikB_i}{\nu_i}} a^2 \left(\frac{1}{2} M_{2ia} - M'_{2ia} \right).$$

$$F_{3,oa} = M_{1oa} + \sqrt{\frac{ikB_o}{\nu_o}} a^2 \left(\frac{1}{2} M_{1oa} - M'_{1oa} \right), \quad F_{4,oa} = M_{2oa} - \sqrt{\frac{ikB_o}{\nu_o}} a^2 \left(\frac{1}{2} M_{2oa} - M'_{2oa} \right),$$

$$k_{1,i} = 2k^2a^2 + \frac{\omega + ikU_{z,i}(a)}{\nu_i} a^2, \qquad k_{1,o} = 2k^2a^2 + \frac{\omega + ikU_{z,o}(a)}{\nu_o} a^2 .$$

$$G_{1,i} = \varepsilon \left[2ik^2a^2 I'_1(ka) + \frac{B_i a^3}{\nu_i}\left(2I_1(ka) - ka\, I_0(ka)\right) - \frac{\omega + ikA_i}{i\nu_i} a^2 I_0(ka) - \frac{ik\sigma}{\mu_i \omega}\left(1 - k^2a^2\right) I_1(ka) \right]$$

$$G_{3,i} = \varepsilon \left[2ika\left(M_{1ia} + \sqrt{\frac{ikB_i}{\nu_i}} a^2 \left(M_{1ia} - 2M'_{1ia}\right)\right) + 4\frac{B_i a^3}{\nu_i} M_{1ia} - \frac{ik\sigma}{\mu_i \omega}\left(1 - k^2a^2\right) M_{1ia} \right]$$

$$G_{4,i} = \varepsilon \left[2ika\left(M_{2ia} - \sqrt{\frac{ikB_i}{\nu_i}} a^2 \left(M_{2ia} - 2M'_{2ia}\right)\right) + 4\frac{B_i a^3}{\nu_i} M_{2ia} - \frac{ik\sigma}{\mu_i \omega}\left(1 - k^2a^2\right) M_{2ia} \right]$$

$$G_{1,o} = 2ik^2a^2 I'_1(ka) + \frac{B_o a^3}{\nu_o}\left(2I_1(ka) - ka\, I_0(ka)\right) - \frac{\omega + ikA_o}{i\nu_o} a^2 I_0(ka)$$

$$G_{2,o} = 2ik^2a^2 \frac{K'_1(ka)}{K_0(ka)} + \frac{B_o a^3}{\nu_o}\left(2\frac{K_1(ka)}{K_0(ka)} + ka \right) + \frac{\omega + ikA_o}{i\nu_o} a^2$$

$$G_{3,o} = 2ika\left(M_{1oa} + \sqrt{\frac{ikB_o}{\nu_o}} a^2 \left(M_{1oa} - 2M'_{1oa}\right)\right) + 4\frac{B_o a^3}{\nu_o} M_{1oa}$$

$$G_{4,o} = 2ika\left(M_{2oa} - \sqrt{\frac{ikB_o}{\nu_o}} a^2 \left(M_{2oa} - 2M'_{2oa}\right)\right) + 4\frac{B_o a^3}{\nu_o} M_{2oa} .$$

The result of our present analysis is therefore a determinantal equation, analogous to what Tomotika obtained. Due to the different base flow case treated here, however, in contrast to [4], the determinant consists of seven components rather than of only four. Also, since we treat a confined flow here, in contrast to the unbounded flow analysed by Tomotika, the determinantal equations are not readily comparable.

4. Analysis of the dispersion relation

Subsequently, we analyse the dispersion relation (51), to some degree following the lines of Tomotika [4]. We first attempt to evaluate (51) in full generality, and then look at the behaviour of the system under the assumption that inertia plays a far smaller role than viscosity and capillarity. This means that, in the second step, we get rid of all terms exhibiting the densities of the two fluids.

For starting, we take as known for the flow problem at hand the two volume flow rates $\dot{V}_i$ and $\dot{V}_o$, and the fluid properties such as the dynamic viscosities μ_i and μ_o and densities ρ_i and ρ_o. Also we take as known the inner radius R of the tube. This enables us to calculate the radius of the filament using equation (8), so that all geometrical parameters of the flow field are known then. Furthermore, we set values of the wave number ka between 0 and 1 first and then determine the complex value of the complex frequency ω, which satisfy Eq. (51). This puts out the oscillation frequency and the damping or growth rates under the disturbance with ka.

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